A1. *Let $p \geq 3$ be a prime.*

5pt (a) *Prove that a nonabelian semidirect product $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$ exists and is unique up to isomorphism.*

Solution. For $p \geq 3$, the group $\mathrm{Aut}(\mathbb{Z}_{p^2})$ is cyclic of order $p^2 - p = p(p-1)$. It contains a single subgroup L of order p; so, there is a nontrivial homomorphism $\mathbb{Z}_p \longrightarrow \mathrm{Aut}(\mathbb{Z}_{p^2})$, which defines a nonabelian semidirect product $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$. Any two such homomorphism have the same image L, so (as we proved) define isomorphic semidirect products.

5pt (b) *Prove that a nonabelian semidirect product $\mathbb{Z}_p^2 \rtimes \mathbb{Z}_p$ exists and is unique up to isomorphism.*

Solution. The group $\mathrm{Aut}(\mathbb{Z}_{p^2})$ is isomorphic to $\mathrm{GL}_2(\mathbb{F}_p)$ of order $(p^2-1)(p^2-p) = p(p-1)^2(p+1)$. By Sylow theorem, $\mathrm{Aut}(\mathbb{Z}_{p^2})$ contains subgroups of order p, and all such subgroups, being Sylow p-subgroups, are conjugate. So, there is a nontrivial homomorphism $\mathbb{Z}_p \longrightarrow \mathrm{Aut}(\mathbb{Z}_p^2)$, which defines a nonabelian semidirect product $\mathbb{Z}_p^2 \rtimes \mathbb{Z}_p$. Since any two such homomorphism have conjugates images, they define isomorphic semidirect products.

5pt (c) *Prove that nonabelian $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_2$ and $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$ exist and are both isomorphic to D_8.*

Solution. In the case $p = 2$, the groups described in (a) and (b) are $G_1 = \big\langle a, b, c \bigm| a^2 = b^2 = c^2 = 1,\ ab = ba,\ cac = b,\ cbc = a \big\rangle$ (the automorphism $a \leftrightarrow b$ of $\big\langle a, b \bigm| a^2 = b^2 = 1,\ ab = ba \big\rangle \cong \mathbb{Z}_2^2$ has order 2) and $G_2 = \big\langle a, b \bigm| a^4 = b^2 = 1,\ bab = a^3 \big\rangle$. Clearly, $G_2 \cong D_8$. An isomorphism between G_1 and D_8 can be established by putting $r = ca$ and $s = a$: indeed, r and s generate G_1, $r^2 = caca = ba = ab$, $r^3 = caab = cb$, $r^4 = (ab)^2 = 1$, so $|r| = 4$, $|s| = |a| = 2$, and $sr = aca = cba = r^3 s$.

10pt **4.5.7.** *Exhibit all Sylow 2-subgroups of S_4 and determine their isomorphism type.*

Solution. Since $|S_4| = 24 = 2^3 \cdot 3$, every Sylow 2 subgroup of S_4 has 8 elements, and $n_2 = 1$ or 3. In S_4, the elements of order 2^k for some k are the identity, six transpositions, three elements of the form $(a,b)(c,d)$ (where a, b, c, d are assumed to be all distinct), and six 4-cycles. Since the total number of such elements is > 8, we have $n_2 = 3$.

Since all Sylow 2-subgroups are conjugate, for every conjugacy class C in S_4 these subgroups must have equal numbers of elements from C. The subgroup V_4, consisting of elements of the form $(a,b)(c,d)$ and the identity, is normal in S_4, thus it is contained in every Sylow 2-subgroup. Hence, each of these subgroups must additionaly contain two transpositions and two 4-cycles, which have to be a 4-cycle and its inverse. A Sylow 2-subgroup cannot contain two transpositions with a common element, since the product of such transpositions is a 3-cycle: $(a,b)(a,c) = (a,c,b)$. So, any Sylow 2-subgroup must contain two transpositions (a,b) and (c,d), with distinct a, b, c, d, three elements from V_4, – $(a,b)(c,d)$, $(a,c)(b,d)$, $(a,d)(b,c)$, the product $(a,b)(a,c)(b,d) = (a,c,b,d)$, and its inverse (d,b,c,a). (We can check that what we get is a group, but this is not neccessary, since we have no other choices anyway.) So, here are the three Sylow 2-subgroups of S_4:

$$\begin{aligned}
P_1 &= \big\{1, (1,2), (3,4), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3), (1,3,2,4), (4,2,3,1)\big\},\\
P_2 &= \big\{1, (1,3), (2,4), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3), (1,2,3,4), (4,3,2,1)\big\},\\
P_3 &= \big\{1, (1,4), (2,3), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3), (1,2,4,3), (3,4,2,1)\big\}.
\end{aligned}$$

The elements that conjugate P_1 to P_2 and P_3 are, for instance, $\rho_2 = (2,3)$ and $\rho_3 = (2,4)$ respectively. In the group P_1, if we put $a = (1,3,2,4)$ and $b = (1,2)$, we have $\langle a \rangle \cap \langle b \rangle = 1$, $|\langle a \rangle| \cdot |\langle b \rangle| = 8 = |P_1|$, $bab^{-1} = (2,3,1,4) = a^{-1}$, so $P_1 \cong D_8$, and so $P_2, P_3 \cong D_8$.

5pt **4.5.13.** *Prove that every group of order 56 has a normal Sylow p-subgroup for some p.*

Solution. We have $56 = 2^3 \cdot 7$. If $n_7 \neq 1$, then $n_7 = 8$, and then G contains $8 \cdot 6 = 48$ elements of order 7. The remaining $56 - 48 = 8$ elements may form only one subgroup of order 8 (the Sylow 2-subgroup of G), so $n_2 = 1$.

10pt **4.5.17, 6.2.15.** (a) *Prove that if $|G| = 105$ then G has a normal Sylow 5 subgroup and a normal Sylow 7-subgroup.*

Solution. We have $105 = 3 \cdot 5 \cdot 7$, so G is a group of order "pqr". n_5 divides 21 and equals 1 modulo 5, so it is either 1 or 21; if $n_5 = 21$, then G has $21 \cdot 4 = 84$ elements of order 5. n_7 divides 15 and equals 1 modulo 7, so it is either 1 or 15; if $n_7 = 15$, then G has $15 \cdot 6 = 90$ elements of order 7. Hence, it is impossible that both $n_5 = 21$ and $n_7 = 15$, thus we have $n_5 = 1$ or $n_7 = 1$ (or both).

Let $P \in \mathrm{Syl}_3(G)$, $Q \in \mathrm{Syl}_5(G)$, and $R \in \mathrm{Syl}_7(G)$, so that $P \cong \mathbb{Z}_3$, $Q \cong \mathbb{Z}_5$, and $R \cong \mathbb{Z}_7$. Since at least one of Q, R is normal, $H = QR$ is a subgroup of order 35. Since $7 \neq 1 \bmod 5$, $H = R \times Q$. Since $|G : H| = 3$, the minimal prime divisor of 105, we have that H is normal in G. Since R and Q are characteristic subgroups of H, they are normal in G.

10pt (b) *Find all (up to isomorphism) groups of order* 105*.*

Solution. The only abelian group of this order is $\mathbb{Z}_{105}$. Assume that G is nonabelian. We have $\mathrm{Aut}(H) \cong \mathbb{Z}_5^* \times \mathbb{Z}_7^* \cong \mathbb{Z}_4 \times \mathbb{Z}_6$, which contains a single subgroup isomorphic to $\mathbb{Z}_3$, so there is a unique (up to isomorphism) nontrivial semidirect product $\mathbb{Z}_{35} \rtimes \mathbb{Z}_3$. An element of order 3 in $\mathbb{Z}_{35}^*$ is one that equals 2 (or 4) modulo 7 and 1 modulo 5, namely, 16. Hence, G has the presentation $\langle a, b \mid a^{35} = b^3 = 1,\ bab^{-1} = a^{16} \rangle$.

10pt **A2.** *Prove that all groups of order p^2q^2, where p and q are prime, are solvable.*

Solution. Let $|G| = p^2q^2$ where p, q are primes. If $p = q$, then G is a p group and is solvable; so, assume that $p < q$. If $n_q \neq 1$, then since $n_q = 1 \bmod q$ we have $n_q > q > p$, and $n_q \mid p^2$, so $n_q = p^2$; this implies that $p^2 = 1 \bmod q$, so $q \mid (p-1)(p+1)$, which only holds for $p = 2$, $q = 3$; but any group of order 36 is solvable. If $n_q = 1$, then G has a normal subgroup Q of order q^2, then $|G/Q| = p^2$, so that both Q and G/Q are abelian, and G is solvable.