Solutions to Midterm **Math 5590H**

10% **1.** *If two subgroups $H, K \le G$ have coprime orders (i.e.* $\gcd(|H|,|K|) = 1$*) prove that* $|HK| = |H| \cdot |K|$.

Solution. Since $|H \cap K|$ divides both $|H|$ and $|K|$, we have $|H \cap K| = 1$. So, $|HK| = |H| \cdot |K|/|H \cap K| = |H| \cdot |K|$.

20% **2.** *Let $H \le G$, $|G : H| = n$, and assume that H contains no nontrivial (i.e. $\neq 1$) subgroups that are normal in G. Prove that G is isomorphic to a subgroup of S_n.*

Solution. The action of G by left multiplication on the set G/H of left cosets of H defines a nontrivial homomorphism $\varphi\colon G \longrightarrow S_n$. $\ker\varphi$ is a normal subgroup of G that is contained in the stabilizer of H under this action, which is H itself. So, $\ker\varphi = 1$ and φ is an isomorphism.

20% **3.** *Let H be a subgroup of finite index in a group G. Prove that the number of subgroups conjugate to H divides $|G : H|$.*

Solution. G acts on the set of subsets of G by conjugations, and the conjugacy class $\mathcal{H}$ of H is the orbit of H under this action. The stabilizer of H under this action is $N_G(H)$, which contains H. Thus, $|\mathcal{H}| = |G : N_G(H)|$, which divides $|G : H|$ since $|G : H| = |G : N_G(H)| \cdot |N_G(H) : H|$.

15% **4.** *Determine whether the groups $\mathbb{Z}_{36} \times \mathbb{Z}_{12} \times \mathbb{Z}_{10}$ and $\mathbb{Z}_{60} \times \mathbb{Z}_{18} \times \mathbb{Z}_4$ are isomorphic.*

Solution. By the Chinese remainder theorem, $\mathbb{Z}_{36} \times \mathbb{Z}_{12} \times \mathbb{Z}_{10} \cong \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_2$ and $\mathbb{Z}_{60} \times \mathbb{Z}_{18} \times \mathbb{Z}_4 \cong \mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ (these are the elementary divisors decompositions of these groups), which are clealry isomorphic.

5. *Let* $G = \langle a, b, c \mid a^5 = b^5 = c^4 = 1,\ ab = ba,\ ca = bc,\ cb = a^4c \rangle$.

15% (a) *Represent G as a semidirect product of abelian groups and find the factors of a (any) composition series of G.*

Solution. Let $H = \langle a, b\rangle$ and $K = \langle c\rangle$. Then $H \trianglelefteq G$ since conjugation by c maps a and b into H. The action of c by conjugation on H has order 4: $a \mapsto b \mapsto a^{-1} \mapsto b^{-1} \mapsto a$, which guarantees that $G = H \rtimes K$, $H \cong \mathbb{Z}_5^2$, $K \cong \mathbb{Z}_4$. The factors of (any) composition series for G are $\mathbb{Z}_5$, $\mathbb{Z}_5$, $\mathbb{Z}_2$ and $\mathbb{Z}_2$.

10% (b) *Represent G is a quotient group F/N of a free group F.*

Solution. We have $G \cong F/N$ where $F = \langle a, b, c\rangle$ and N is the minimal normal subgroup of F containing the elements a^5, b^5, c^4, $aba^{-1}b^{-1}$, $cac^{-1}b^{-1}$, and $cbc^{-1}a^{-4}$.

20% **6.** *If P is a Sylow p-subgroup of a finite group G and $N_G(P)$ is normal in G, prove that G has a unique Sylow p-subgroup.*

Solution. P is a Sylow subgroup of $N_G(P)$ and is normal in $N_G(P)$, so $P \operatorname{char} N_G(P)$. If also $N_G(P) \trianglelefteq G$, we have $P \trianglelefteq G$.

20% **7.** *Find all, up to isomorphism, groups of order* $155 = 31 \cdot 5$.

Solution. Let G be a group with $|G| = 155$. We have $n_{31} = 1 \bmod 31$ and $n_{31} \mid 5$, so $n_{31} = 1$. Let P be the (only) Sylow 31-subgroup of G, $P = \langle a\rangle$, and Q be a Sylow 5-subgroup of G, $Q = \langle b\rangle$; then $G = P \rtimes Q$. One such group is $P \times Q \cong \mathbb{Z}_{31} \times \mathbb{Z}_5 \cong \mathbb{Z}_{155}$. Since $31 = 1 \bmod 5$, there also exist a nonabelian semidrect product: the element 2 has order 5 in $\mathbb{Z}_{31}^*$, so G has presentation $\langle a, b \mid a^{31} = b^5 = 1,\ bab^{-1} = a^2\rangle$.

20% **8.** *Prove that all groups of order* $1452 = 2^2 \cdot 3 \cdot 11^2$ *are solvable.*

Solution. We have $n_{11} = 1 \bmod 11$ and $n_{11} \mid 2^2 \cdot 3 = 12$, so either $n_{11} = 1$ or $n_{11} = 12$.

If $n_{11} = 1$, the the Sylow 11-subgroup P of G is normal in G, $|P| = 11^2$, so P is abelian, and $|G/P| = 12$, so G/P is solvable. Hence, G is solvable in this case.

If $n_{11} = 12$, we have a nontrivial homomorphism $\varphi\colon G \longrightarrow S_{12}$ (induced by the action of G on $\mathrm{Syl}_{11}(G)$). Let $K = \varphi(G)$ and $H = \ker\varphi$, so that $H \triangleleft G$, $K \cong G/H$, $|G| = |H| \cdot |K|$. Since $|K| \mid |S_{12}| = 12!$ and $12!$ is not divisible by 11^2, H is nontrivial. Hence, G is not simple.

To prove that G is solvable we need to show that both H and K are solvable; it suffices to check that every nonabelian group N with $|N| < |G|$ and $|N| \mid |G|$ is not simple too. For $|N| = 2 \cdot 3 \cdot 11^2$ we have

$n_{11}(N) = 1$, so N is not simple. For $|N| = 2^2 \cdot 3 \cdot 11 = 132$, some $n_p(N)$ must be equal to 1 by counting elements: if $n_{11} = 12$ and $n_3 = 4$, then, since $12 \cdot 10 + 4 \cdot 2 = 128$, N may have only one subgroup of order 4. All other possible orders are of one of the "standard" types p, p^2, p^2q, pq^2, or pqr, and the groups of such orders are all solvable.