Math 5591H **Final exam**

You may use any fact proven in class, in the textbook, or in homework.

15% **1.** Let θ be such that $\cos\theta = 5/7$; prove that the angle $\theta/5$ is not constructible with ruler and compass. (You may use the identity $\cos(5x) = 16\cos^5 x - 20\cos^3 x + 5\cos x$.)

15% **2.** Let F be a finite field and let $f \in F[x]$ be a product of k irreducible polynomials of degrees $n_1, \ldots, n_k$. Find $\mathrm{Gal}(f/F)$. (*Hint:* What is the Galois group of a finite field?)

20% **3.** Let F be a field with $\mathrm{char}\, F \neq 2$, let $f \in F[x]$ be a separable polynomial, let $G = \mathrm{Gal}(f/F)$. Let $\widetilde{G} = \mathrm{Gal}(f(x^2)/F)$; prove that there is an exact sequence $1 \longrightarrow \mathbb{Z}_2^d \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$ (in other words, $\widetilde{G}$ has a normal subgroup N isomorphic to $\mathbb{Z}_2^d$ such that $\widetilde{G}/N \cong G$) for some $d \geq 0$.

20% **4.** An irreducible quartic $f \in \mathbb{Q}[x]$ has two real and two non-real complex roots and its cubic resolvent has a single root in $\mathbb{Q}$. Prove that $\mathrm{Gal}(f/\mathbb{Q}) \cong D_8$.

40% **5.** Let $\alpha = \sqrt{(2+\sqrt{2})(3+\sqrt{3})} \in \mathbb{R}$ and let $K = \mathbb{Q}(\alpha)$. Take it for granted that $\alpha \notin \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

(a) Prove that $\deg_{\mathbb{Q}}(\alpha^2) = 4$ and deduce that $\sqrt{2}, \sqrt{3} \in K$.

(b) Find the degree and all the conjugates of α over $\mathbb{Q}$.

(c) Show that the extension $K/\mathbb{Q}$ is normal. (*Hint:* For $\beta = \sqrt{(2-\sqrt{2})(3+\sqrt{3})}$ we have $\alpha\beta = \sqrt{2}(3+\sqrt{3}) \in K$.)

(d) Let $G = \mathrm{Gal}(K/\mathbb{Q})$. Prove that there exists $\varphi \in G$ such that $\varphi(\sqrt{2}) = -\sqrt{2}$ and $\varphi(\sqrt{3}) = \sqrt{3}$. Prove that $|\varphi| = 4$. (*Hint:* Show that $\varphi(\alpha) = \beta$ or $-\beta$.)

(e) Find two more automorphisms of K of order 4 and deduce that $G \cong Q_8$ (the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$).

(f) Draw the lattice (the diagram) of all the subfields of K.

Good luck! and have a nice vacation