Homework 1 **Math 5591H**

Due by Wednesday, January 21

Let R be a ring.

If R is an integral domain and M is an R-module, we proved that $\operatorname{Tor}(M) = \{u \in M : au = 0 \text{ for some nonzero } a \in R\}$ is a submodule of M.

5pt **A1.** If R is an integral domain and M is an R-module, prove that the quotient module $\widetilde{M} = M/\operatorname{Tor}(M)$ is torsion-free, that is, $\operatorname{Tor}(\widetilde{M}) = 0$.

10pt **A2.** Let M be a left R-module, A be an abelian group, H be the set of homomorphisms from the additive group of M to A (that is, mappings $f\colon M \longrightarrow A$ such that $f(u+v) = f(u) + f(v)$ for all $u, v \in M$). Let's define the addition on H by $(f+g)(u) = f(u) + g(u)$ and the "multiplication by scalars" by $(af)(u) = f(au)$, $f, g \in H$, $a \in R$, $u \in M$. Prove that these operations turn H into a RIGHT R-module.

5pt **10.3.7.** Let M be an R-module and N be its submodule. If both N and M/N are finitely generated, prove that M is finitely generated. More exactly: if N is generated by n elements and M/N is generated by m elements, prove that M is generated by $n+m$ elements.

If M is a left R-module and S is a subset of R, the annihilator of S is $\operatorname{Ann}(S) = \{u \in M : au = 0 \text{ for all } a \in P\}$.

A3. Let M be an R-module and let I, J be right ideals in R.

5pt (a) Prove that $\operatorname{Ann}(I)$ (and $\operatorname{Ann}(J)$) is a submodule of M.

5pt (b) Prove that $\operatorname{Ann}(I+J) = \operatorname{Ann}(I) \cap \operatorname{Ann}(J)$.

5pt (c) Prove that $\operatorname{Ann}(I \cap J) \supseteq \operatorname{Ann}(I) + \operatorname{Ann}(J)$.

5pt (d) Give an example where the inclusion in (c) is strict, $\operatorname{Ann}(I \cap J) \neq \operatorname{Ann}(I) + \operatorname{Ann}(J)$. (*Hint:* This is not easy. Try $R = F[x,y]$, $M = R/(xy)$, $I = (x)$, $J = (y)$. Alternatively, take $R = M = C([0,2])$, $I = \{f : f|_{[0,1]} = 0\}$, $J = \{f : f|_{[1,2]} = 0\}$.)

5pt (e) If R is commutative and unital and I, J are comaximal (that is, $I + J = (1)$), prove that $\operatorname{Ann}(I \cap J) = \operatorname{Ann}(I) + \operatorname{Ann}(J)$.

5pt **10.2.9.** Let R be a commutative unital ring and M be an R-module. Prove that $\operatorname{Hom}_R(R, M)$ and M are isomorphic as R-modules.

A is *an R-algebra* if A is a ring and an R-module so that $a(\alpha\beta) = (a\alpha)\beta = \alpha(a\beta)$ for all $a \in R$, $\alpha, \beta \in A$. In the textbook algebras are defined in a different way: A is an R-algebra if A is a ring with a ring homomorphism $R \longrightarrow Z(A)$. The following problem demonstrates that the two definitions are pretty close.

5pt **A4.** (a) Let A be a ring and $\varphi\colon R \longrightarrow A$ be a ring homomorphism such that $\varphi(R) \subseteq Z(A)$ (where $Z(A)$ is the center of A). Prove that under the "multiplication by scalars" defined by $a\alpha = \varphi(a)\alpha$ for $a \in R$, $\alpha \in A$, A is an R algebra.

5pt (b) Let A be a unital R-algebra, that is, with $1 \in A$. Find a homomorphism $\varphi\colon R \longrightarrow A$ such that $a\alpha = \varphi(a)\alpha$ for all $a \in R$, $\alpha \in A$, and show that $\varphi(R) \subseteq Z(A)$.