Homework 11 **Math 5591H**

Due by Tuesday, April 16

Cf. 14.2.28. Let $f \in F[x]$ be an irreducible polynomial of degree n over a field F, let α be a root of f, and let K/F be a normal extension. Show that f splits over K into a product of irreducible polynomials of the same degree $d = [K(\alpha) : K]$. (You may assume that f is separable and K/F is finite and separable.) (*Hint:* If f is separable, let L be a splitting field of f over K, let L/F be the splitting field of f over K If $\alpha, \beta \in L$ are two roots of f, prove that there is $\varphi \in \mathrm{Aut}(L/F)$ with $\varphi(\alpha) = \beta$, and show that $\varphi(m_{\alpha,K}) = m_{\beta,K}$.)

Cf. 14.6.20. Let K be the splittig field of $f(x) = (x^3 - 2)(x^3 - 3) \in \mathbb{Q}[x]$, let $G = \mathrm{Gal}(K/\mathbb{Q})$. Let $\alpha = \sqrt[3]{2}$, $\beta = \sqrt[3]{3}$, $\omega = e^{2\pi i/3}$.

(a) Consider K as the composite $\mathbb{Q}(\alpha, \omega)\mathbb{Q}(\beta)$ and represent G as a semidirect product of S_3 and $\mathbb{Z}_3$. (Don't specify the homomorphism that defines the semidirect product, if you don't want to.)

(b) Consider K as the composite $\mathbb{Q}(\alpha, \omega)\mathbb{Q}(\beta, \omega)$ and represent G as a "relative direct product" $S_3 \times_{\mathbb{Z}_2} S_3$.

(c) Find all the subfields of K that contain $N = \mathbb{Q}(\omega)$.

A1. Let $\alpha = \sqrt{2} + \sqrt{3} + \sqrt{5}$.

(a) Find all the conjugates of α over $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and find the minimal polynomial $m_{\alpha,L}$.

(b) Find all the conjugates of α over $N = \mathbb{Q}(\sqrt{2})$ and find the minimal polynomial $m_{\alpha,N}$.

(c) Find the minimal polynomial $m_{\alpha,\mathbb{Q}}$.

(d) Prove that α is a primitive element of $\mathbb{Q}\big(\sqrt{2}, \sqrt{3}, \sqrt{5}\big)/\mathbb{Q}$.

A2. Let K be a cubic extension $\mathbb{Q}\big(\sqrt[3]{D}\big)$ of $\mathbb{Q}$. Obtain the formula for the norm $N_{K/\mathbb{Q}}(\alpha)$ of the element $\alpha = a + b\sqrt[3]{D} + c\sqrt[3]{D^2}$, $a, b, c \in \mathbb{Q}$, of K.

A3. Prove the following:

(a) If K/F is a p-extension and L/F is a subextension of K/F, then both K/L and L/F are p-extension.

(b) If L_1/F and L_2/F are p-subextensions of an extension K/F, then their composite L_1L_2/F is a p-extension.

(c) If K/L and L/F are p-extensions, then K/F is also a p-extension.

14.7.12. Let K be a Galois closure of a finite extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ and let $G = \mathrm{Gal}(K/\mathbb{Q})$. For every prime p dividing $|G|$, prove that there exists a subfield L of K such that $[K : L] = p$ and $K = L(\alpha)$.