Homework 12 **Math 5591H**

Due by Tuesday, April 21

10pt **A1.** Let F be a field with $\operatorname{char} F \neq 3$, let $d \in F$ be such that $x^3 - d$ is irreducible, let $K = F\big(\sqrt[3]{d}\big)$. Obtain the formula for the norm $N_{K/F}(\alpha)$ of the element $\alpha = a+b\sqrt[3]{d}+c\sqrt[3]{d^2}$, $a, b, c \in F$, of K.

5pt **A2.** (a) Let $\alpha = \sqrt{2\sqrt{2} + \sqrt{5\sqrt{3} + 3\sqrt{5}} + \sqrt{6}}$, let $K/\mathbb{Q}$ be the Galois closure of $\mathbb{Q}(\alpha)/\mathbb{Q}$. Prove that $[K : \mathbb{Q}] = 2^n$ for some $n \in \mathbb{N}$.

10pt (b) Let $\alpha = \sqrt[3]{2\sqrt[3]{2} + \sqrt[3]{5\sqrt[3]{3} + 3\sqrt[3]{5}} + \sqrt[3]{6}}$, let $K/\mathbb{Q}$ be the Galois closure of $\mathbb{Q}(\alpha)/\mathbb{Q}$. Prove that $[K : \mathbb{Q}] = 2 \cdot 3^n$ for some $n \in \mathbb{N}$. (You may assume without proof that K contains an irrational cubic root of a rational number, say, $\sqrt[3]{2}$.) (*Hint:* Consider K as an extension of $L = \mathbb{Q}(\omega)$ where $\omega = e^{2\pi i/3}$.)

14.4.6. Let p be a prime, let $K = \mathbb{F}_p(x, y)$ (the field of rational functions over $\mathbb{F}_p$ in the variables x and y), let $F = \mathbb{F}_p(x^p, y^p) \subset K$.

5pt (a) Prove that $\deg_F x = \deg_F y = p$ and $[K : F] = p^2$.

5pt (b) Prove that the extension K/F is normal.

5pt (c) Prove that K is not a simple extension of F: there is no $\alpha \in K$ such that $K = F(\alpha)$.

5pt (d) Explain why the result of (c) doesn't contradict "the theorem on the primitive element".

5pt **A3.** Let θ be such that $\cos\theta = 5/7$; prove that the angle $\theta/5$ is not constructible with ruler and compass. (You may use the identity $\cos(5x) = 16\cos^5 x - 20\cos^3 x + 5\cos x$.)

5pt **13.3.1,4.** Prove that it is impossible to "construct with ruler and compass" the regular 7- and 9-gons.

10pt **A4.** Prove that there are no nontrivial automorphisms of $\mathbb{R}$. (*Hint:* Let $\varphi \in \operatorname{Aut}(\mathbb{R})$; prove that φ fixes $\mathbb{Q}$. Prove that φ maps positive numbers to positive numbers and deduce that φ preserves the order on $\mathbb{R}$.)