Homework 2 **Math 5591H**

Due by Tuesday, January 27

10.5.1(d,e). Suppose that

$$\begin{array}{ccccc} A & \xrightarrow{\psi} & B & \xrightarrow{\varphi} & C \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow \\ A' & \xrightarrow{\psi'} & B' & \xrightarrow{\varphi'} & C' \end{array}$$

is a commutative diagram of module homomorphisms with exact rows.

10pt (d) If β is injective and α and φ are surjective, prove that γ is injective.

10pt (e) If β is surjective and γ and ψ' are injective, prove that α is surjective.

A1. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\psi} & B & \xrightarrow{\varphi} & C & \longrightarrow & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\ 0 & \longrightarrow & A' & \xrightarrow{\psi'} & B' & \xrightarrow{\varphi'} & C' & \longrightarrow & 0 \end{array}$$

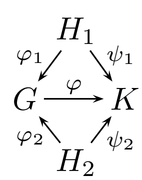
be a commutative diagram of module homomorphisms with exact rows. Define a mapping $\delta\colon \ker\gamma \longrightarrow \operatorname{coker}\alpha$ in the follwing way: for $c \in \ker(\gamma)$ choose $b \in \varphi^{-1}(c)$ and put $b' = \beta(b)$; then $\varphi'(b') = \gamma(c) = 0$, so $b' = \psi'(a')$ for some $a' \in A'$. Now put $\delta(c) = a' \bmod \alpha(A)$.

5pt (a) Prove that δ is well defined (i.e. doesn't depend on the choice of b) and is a homomorphism.

10pt (b) Prove that $\ker(\delta) = \varphi(\ker(\beta))$.

5pt **A2.** Consider the category where objects are integers and a morphism (an arrow) from object n to object m exists, and is unique, iff $n \mid m$. (Notice that in this category objects are not assumed to be sets and morphisms are not mappings!) The composition of two morphisms exists and is defined uniquely, since if $n \mid m$ and $m \mid k$ then $n \mid k$. What objects in this category are isomorphic? Are there universal repelling and/or attracting objects?

10pt **A3.** Let H_1 and H_2 be two groups. Consider the category of groups G with homomorphisms $H_1, H_2 \longrightarrow G$: the objects are triplets $(G, \varphi_1, \varphi_2)$ where G is a group and $\varphi_i\colon H_i \longrightarrow G$, $i = 1, 2$, are group homomorphisms; the morphisms $(G, \varphi_1, \varphi_2) \longrightarrow (K, \psi_1, \psi_2)$ are homomorphisms $\varphi\colon G \longrightarrow K$ for which the diagram



is commutative. Prove that in this category the universal repelling object is *the free product* $H_1 * H_2$ of H_1 and H_2, defined in the following way. The elements of $H_1 * H_2$ are "alternating" words of the form $a_1 \cdots a_k$ where for all i, $a_i \in H_1 \setminus \{1\}$ or $H_2 \setminus \{1\}$ with $a_{i+1} \in H_2$ if $a_i \in H_1$ and $a_{i+1} \in H_1$ if $a_i \in H_2$. (It is assumed that H_1 and H_2 are disjoint; otherwise we replace them by their disjoint copies.) The operation in $H_1 * H_2$ is concatenation of such words with natural reduction: if a subword bc with both $b, c \in H_1$ or $b, c \in H_2$

occurs, it is replaced by the element bc of this group, and in the case $bc = 1$ it is removed and the process of reduction continues.

10.3.12. Let R be a commutative ring and let A, B, and M be R-modules. Prove the following isomorphisms of R-modules:

5pt (a) $\mathrm{Hom}_R(A \oplus B, M) \cong \mathrm{Hom}_R(A, M) \oplus \mathrm{Hom}_R(B, M)$.

5pt (b) $\mathrm{Hom}_R(M, A \oplus B) \cong \mathrm{Hom}_R(M, A) \oplus \mathrm{Hom}_R(M, B)$.

An element e of a ring R is said to be *idempotent* if $e^2 = e$. An element of R is said to be *central* if it is contained in the center of R, that is, commutes with every element of R.

5pt **10.3.15.** If e is a central idempotent in a unital ring R and M is an R-module, prove that $eM = \mathrm{Ann}(1-e)$, $(1-e)M = \mathrm{Ann}(e)$, and $M = eM \oplus (1-e)M$.

5pt **11.2.11(a,b).** Let R be a unital ring and let φ be an endomorphism of an R-module M such that $\varphi^2 = \varphi$. Prove that $M = \varphi(M) \oplus \ker \varphi$. (*Hint:* You can prove this directly; however it would be nice to interpret M as an $R[\varphi]$-module and use the result of Exercise 10.3.15.)