

### Homework 3

Math 5591H

Due by Tuesday, February 3

In all problems, all modules are assumed to be over a commutative unital ring  $R$ .

- 5pt **10.4.14,15.** (a) Prove that for any modules  $M_1, M_2$ , and  $N$ ,  $(M_1 \oplus M_2) \otimes N \cong (M_1 \otimes N) \oplus (M_2 \otimes N)$ , with  $(u_1, u_2) \otimes v \leftrightarrow ((u_1 \otimes v), (u_2 \otimes v))$ . (*Hint:* Use the universal properties of  $\otimes$  and  $\oplus$ . If you manage to solve (b), it is ok to say that (a) is just a special case.)
- 5pt (b) Prove that for any family  $(M_\alpha)_{\alpha \in \Lambda}$  of modules and any module  $N$ ,  $(\bigoplus_{\alpha \in \Lambda} M_\alpha) \otimes N \cong \bigoplus_{\alpha \in \Lambda} (M_\alpha \otimes N)$ .
- 10pt (c) Give an example where  $(\prod_{\alpha \in \Lambda} M_\alpha) \otimes N \not\cong \prod_{\alpha \in \Lambda} (M_\alpha \otimes N)$ . (*Hint:* With  $R = \mathbb{Z}$ , find a sequence  $M_1, M_2, \dots$  of torsion  $\mathbb{Z}$ -modules such that  $\prod_{i=1}^{\infty} M_i$  is not a torsion module and take  $N = \mathbb{Q}$ .)

**10.4.10.** Let  $N \cong R^n$  be a free module with basis  $\{e_1, \dots, e_n\}$ .

- 5pt (a) Let  $M$  be a nonzero  $R$ -module. Show that every element of  $M \otimes N$  can be written uniquely in the form  $\sum_{i=1}^n m_i \otimes e_i$  where  $m_i \in M$ . Deduce that if  $\sum_{i=1}^n m_i \otimes e_i = 0$  then  $m_i = 0$  for all  $i$ . (*Hint:* Use 10.4.14.)
- 5pt (b) If  $\{e_1, \dots, e_n\}$  isn't a basis but only a maximal linearly independent subset of  $N$ , show that the second assertion in (a) needs not be true. (*Hint:* There is an example with  $n = 1$  and  $R = \mathbb{Z}$ .)

- 5pt **10.4.11.** Let  $\{e_1, e_2\}$  be a basis in  $V = \mathbb{R}^2$ . Show that the element  $e_1 \otimes e_2 + e_2 \otimes e_1$  is not (cannot be written as) a simple tensor in  $V \otimes_{\mathbb{R}} V$ . (*Hint:*  $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$  is a basis of  $V \otimes V$ . Determine the coordinates of the simple tensors with respect to this basis.)

**10.4.16.** Let  $I, J$  be ideals in  $R$ .

- 5pt (a) Prove that every element of  $R/I \otimes R/J$  can be written as a simple tensor of the form  $(1 \bmod I) \otimes (r \bmod J)$ .
- 5pt (b) Prove that there is an isomorphism  $R/I \otimes R/J \cong R/(I + J)$  with  $(r \bmod I) \otimes (r' \bmod J) \mapsto rr' \bmod (I + J)$ ,  $r, r' \in R$ .

**10.4.21.** Let  $I$  and  $J$  be ideals in  $R$ .

- 5pt (a) Show that there exists a surjective homomorphism  $\varphi: I \otimes_R J \rightarrow IJ$  with  $\varphi(i \otimes j) = ij$  for all  $i \in I, j \in J$ .
- 10pt (b) Give an example to show that  $\varphi$  may not be injective.

If  $A$  and  $B$  are  $R$ -algebras, then  $A \otimes_R B$  is also an  $R$ -algebra by  $(\alpha_1 \otimes \beta_1)(\alpha_2 \otimes \beta_2) = (\alpha_1 \alpha_2) \otimes (\beta_1 \beta_2)$ .

- 5pt **10.4.25.** Let  $S$  be an  $R$ -algebra; prove that  $S \otimes_R R[x] \cong S[x]$  as  $R$ -algebras. (*Hint:* As an  $R$ -module,  $R[x]$  is free with basis  $\{1, x, x^2, \dots\}$ , so the module  $S \otimes_R R[x]$  is easy to find. You only have to check the multiplication.)

- 5pt **A1.** Prove that  $R[x] \otimes_R R[y] \cong R[x, y]$  as  $R$ -algebras.