Homework 4 **Math 5591H**

Due by Tuesday, February 10

In all problems, all modules are assumed to be over a commutative unital ring R.

A1. The following example show that the tensor product of torsion-free modules may have a torsion: Let $R = \mathbb{Z}[x,y]$, let I be the ideal (x,y) in R, considered as an R-module, and let $w = x \otimes y - y \otimes x \in I \otimes I$.

5pt (a) Prove that w is a torsion element of $I \otimes I$.

5pt (b) Prove that $w \neq 0$. (*Hint:* Prove that there is a homomorphism $\varphi\colon I \otimes I \longrightarrow \mathbb{Z}$ such that $\varphi\Big((a_1x + a_2y + a_3xy + \cdots) \otimes (b_1x + b_2y + b_3xy + \cdots)\Big) = a_1b_2$ and show that $\varphi(w) \neq 0$.)

10pt **A2.** If a short exact sequence $0 \longrightarrow N \xrightarrow{\varphi} M \xrightarrow{\psi} K \longrightarrow 0$ of modules splits from the left (that is, there is a homomorhism $\tau\colon M \longrightarrow N$ such that $\tau\circ\varphi = \mathrm{Id}_N$) prove that $M = N' \oplus K'$ where $N' = \varphi(N)$ is isomorphic to N (under φ) and $K' = \ker\tau$ is isomorphic to K under $\psi|_{K'}$.

10pt **A3.** Let A be a unital R-algebra and M be an A-module. By reducing scalars, consider M as an R-module (with $au = (a1_A)u$). Prove that the R-module homomorphism $\varphi\colon M \longrightarrow A \otimes_R M$ defined by $\varphi(u) = 1_A \otimes u$ is injective and that $1_A \otimes M = \varphi(M)$ is a direct summand in $A \otimes_R M$. (*Hint:* Prove that the exact sequence $0 \longrightarrow M \xrightarrow{\varphi} A \otimes_R M \longrightarrow \Big(A \otimes_R M\Big)/\varphi(M) \longrightarrow 0$ splits from the left.)

A4. If $\varphi_1\colon M_1 \longrightarrow N_1$ and $\varphi_2\colon M_2 \longrightarrow N_2$ are two homomorphisms of R-modules, then the homomorphism $\varphi_1 \otimes \varphi_2\colon M_1 \otimes M_2 \longrightarrow N_1 \otimes N_2$ is defined by $(\varphi_1 \otimes \varphi_2)(u_1 \otimes u_2) = \varphi_1(u_1) \otimes \varphi_2(u_2)$. However, the same notation $\varphi_1 \otimes \varphi_2$ applies to an element of the tensor product $\mathrm{Hom}(M_1, N_1) \otimes \mathrm{Hom}(M_2, N_2)$.

5pt (a) Prove that there is a unique homomorphism

$$\mathrm{Hom}(M_1, N_1) \otimes \mathrm{Hom}(M_2, N_2) \longrightarrow \mathrm{Hom}(M_1 \otimes M_2, N_1 \otimes N_2)$$

that maps $\varphi_1 \otimes \varphi_2$ in the second sense to $\varphi_1 \otimes \varphi_2$ in the first sense.

5pt (b) Find an example where the homomorphism in (a) is not an isomorphism.

A graded R-algebra is an R-algebra A represented as a direct sum $A = \bigoplus_{i=0}^{\infty} A_i$ of its R-submodules such that $A_iA_j \subseteq A_{i+j}$ for all i, j. An element u of A is said to be *homogeneous* if $u \in A_i$ for some i; every element u of A is uniquely representable as a (finite) sum $u = u_0 + \cdots + u_k$ of homogeneous elements, called *the components* of u. A (two-sided) ideal I in A is said to be graded if $I = \bigoplus_{i=0}^{\infty} I_i$ where for every i, $I_i = I \cap A_i$. An ideal I is said to be *homogeneous* if it is generated (as an ideal!) by homogeneous elements of A.

A5. Let A be a graded algebra.

10pt (a) Prove that a (two-sided) ideal I in A is graded iff for every $u \in I$, all homogeneous components of u are also contained in I.

5pt (b) Prove that an ideal I in A is graded iff it is homogeneous.

5pt (c) If I is a graded ideal in A, prove that A/I has a structure of a graded algebra.

5pt **A6.** Let M be an R-module, $u_1, u_2 \in M$, $a_1, a_2, b_1, b_2 \in R$, $v_1 = a_1u_1 + a_2u_2$ and $v_2 = b_1u_1 + b_2u_2$. Find $c \in R$ such that in $\Lambda^2(M)$, $v_1 \wedge v_2 = cu_1 \wedge u_2$.