Homework 5 **Math 5591H**

Due by Tuesday, February 17

In all problems, all modules are assumed to be over a commutative unital ring R.

5pt **10.5.22.** Prove that the tensor product of two flat modules is flat.

If S is a commutative unital R-algebra, K is an R-modue and M is an S-module, the S-module structure on $M \otimes_R K$ is defined by $\alpha(u \otimes v) = (\alpha u) \otimes v$.

10.5.23. Let S be a commutative unital R-algebra.

10pt (a) If K is an R-module and M is an S-module, prove that $M \otimes_R K \cong M \otimes_S (S \otimes_R K)$ as S-modules.

10pt (b) If K is a flat R-module prove that $S \otimes_R K$ is a flat S-module.

5pt **A1.** Let $R = \mathbb{Z}[x, y]$ and let I be the ideal (x, y) in R. Prove that as an R-module, I is torsion-free but not flat.

5pt **10.5.9(b).** Prove that the tensor product of two projective modules is projective. (*Hint:* Use the criterion of projectivity.)

5pt **A2.** Prove that a module P is projective iff it "splits from the right any short exact sequence", that is, iff every epimorphism $\pi \colon M \longrightarrow P$ has a section (a homomorphism $\sigma \colon P \longrightarrow M$ such that $\pi \circ \sigma = \mathrm{Id}_P$).

5pt **A3.** If F is a field, prove that every F-module is flat, projective, and injective.

5pt **A4.** Prove that an integral domain R is an injective R-module iff R is a field.

5pt **10.5.4.** Given modules Q_1 and Q_2, prove that $Q = Q_1 \oplus Q_2$ is injective iff both Q_1 and Q_2 are injective.

10.5.12. Let M_α, $\alpha \in \Lambda$, be a family of R-modules.

5pt (a) Prove that for any module N, $\mathrm{Hom}\big(N, \prod_{\alpha\in\Lambda} M_\alpha\big) \cong \prod_{\alpha\in\Lambda} \mathrm{Hom}(N, M_\alpha)$ as R-modules.

5pt (b) Prove that for any module N, $\mathrm{Hom}\big(\bigoplus_{\alpha\in\Lambda} M_\alpha, N\big) \cong \prod_{\alpha\in\Lambda} \mathrm{Hom}(M_\alpha, N)$ as R-modules. In particular, $\big(\bigoplus_{\alpha\in\Lambda} M_\alpha\big)^* \cong \prod_{\alpha\in\Lambda} M_\alpha^*$.