Homework 6 **Math 5591H**

Due by Tuesday, February 24

In all problems, all modules are assumed to be over a commutative unital ring R.

For a module M and its submodule N we define $\mathrm{Ann}(N) = \{f \in M^* : f_{|N} = 0\}$, which is a submodule of M^* (the kernel of the epimorphism $M^* \longrightarrow N^*$). For a submodule L of M^*, $\mathrm{Ann}(L)$ is the submodule $\{u \in M : f(u) = 0 \text{ for all } f \in L\}$ of M. The following is clear, and you may use it below: if N is a submodule of a module M, then $\mathrm{Ann}(\mathrm{Ann}(N)) \supseteq N$; if N_1 and N_2 are two submodules of M, then $\mathrm{Ann}(N_1 + N_2) = \mathrm{Ann}(N_1) \cap \mathrm{Ann}(N_2)$ and $\mathrm{Ann}(N_1 \cap N_2) \supseteq \mathrm{Ann}(N_1) + \mathrm{Ann}(N_2)$.

5pt **A1.** (a) For a subspace W of a finite dimensional vector space V, prove that $W = \mathrm{Ann}(\mathrm{Ann}(W))$. (*Hint:* Compare dimensions.) Deduce that for two subspaces W_1 and W_2 of V one has $\mathrm{Ann}(W_1) = \mathrm{Ann}(W_2)$ iff $W_1 = W_2$. (Don't prove that $W_1 = W_2$ implies that $\mathrm{Ann}(W_1) = \mathrm{Ann}(W_2)$, this is trivial!)

5pt (b) If W_1 and W_2 are two subspaces of a finite dimensional vector space V, prove that $\mathrm{Ann}(W_1 \cap W_2) = \mathrm{Ann}(W_1) + \mathrm{Ann}(W_2)$. (*Hint:* Use (a).)

5pt **A2.** If M and N are free modules of fintie rank, prove that the natural homomorphism $M^* \otimes N^* \longrightarrow (M \otimes N)^*$, $f \otimes g \mapsto h$ defined by $h(u \otimes v) = f(u)g(v)$, is an isomorphism. (*Hint:* You can show that under this homomorphism, a basis of $M^* \otimes N^*$ is bijectively mapped onto a basis of $(M \otimes N)^*$.)

The rank, $\mathrm{rank}\,\omega$, of a tensor ω is defined as the minimal number of simple tensors that sum to ω.

10pt **A3.** Let R be an integral domain, let $\varphi\colon M \longrightarrow N$ be a homomorphism of two free R-modules of finite rank, and let ω be the tensor corresponding to φ in $N \otimes M^*$. Prove that $\mathrm{rank}\,\varphi \le \mathrm{rank}\,\omega$, and if R is a field, then $\mathrm{rank}\,\varphi = \mathrm{rank}\,\omega$.

5pt **A4.** Let M be a free module of finite rank, and let P be the transition matrix from a basis B to a basis C in M. Prove that the transition matrix from the basis B^* to the basis C^* in M^* is $(P^T)^{-1}$. (B^* and C^* are the dual bases of B and C respectively.)

5pt **11.3.4.** If M is a free module of an infnite rank with (an infnite) basis B, prove that B^* does not generate M^*.

5pt **A5.** (a) If R is an integral domain, prove that every finitely generated R-module has finite rank.

5pt (b) Give an example of a non-finitely generated module of finite rank.

A6. Let R be an ID, F be the field of fractions of R, M be an R-module, V be the F-vector space $F \otimes_R M$, $M^* = \mathrm{Hom}_R(M, R)$ and $V^* = \mathrm{Hom}_F(V, F)$.

10pt (a) For $f \in M^*$, define $\tilde{f} \in V^*$ by $\tilde{f}(\alpha \otimes u) = \alpha f(u)$; prove that $\tilde{f}$ is well defined. Prove that the mapping $f \mapsto \tilde{f}$ is a monomorphism (injective homomorphism) $M^* \longrightarrow V^*$. Deduce that the F-space homomorphism $F \otimes_R M^* \longrightarrow V^*$ defined by $1 \otimes f \mapsto \tilde{f}$ is also injective, and in the case $\mathrm{rank}\,M < \infty$, that $\mathrm{rank}\,M^* \le \mathrm{rank}\,M$.

10pt (b) If M is finitely generated, prove that the image of M^* in V^* spans (generates) V^*; moreover, for any $h \in V^*$ there exist $f \in M^*$ and $d \in R$ such that $dh = \tilde{f}$. Deduce that $F \otimes M^* \cong V^*$ and that $\mathrm{rank}\,M^* = \mathrm{rank}\,M$.

5pt (c) Give an example where for an (infinitely generated) R-module M of finite rank, the image of M^* doesn't span V^* (and so, $\mathrm{rank}\,M^* < \mathrm{rank}\,M$).