Homework 7 **Math 5591H**

Due by Tuesday, March 3

5pt **A1.** If M is a finitely generated module over a PID, prove that M is flat iff M is projective iff M is torsion-free iff M is free.

A2. Give an example of a module M over an integral domain R such that M is torsion-free but not free and

5pt (a) M is finitely generated but R is not a PID.

5pt (b) R is a PID but M is not finitely generated.

A3. Let M be a free $\mathbb{Z}$-module of rank 4, $M_1 \cong M/N_1$ and $M_2 \cong M/N_2$, where N_1 is a subgroup (submodule) of M generated, in some basis $\{u_1, \dots, u_4\}$ of M, by $\{u_1, 2u_2, 6u_3\}$, and N_2 is a subgroup of M generated, in some basis $\{v_1, \dots, v_4\}$ of M, by $\{v_1, 3v_2, 6v_3\}$.

5pt (a) Are the modules N_1 and N_2 isomorphic?

5pt (b) What are the ranks of the modules M_1 and M_2?

5pt (c) Are the modules M_1 and M_2 isomorphic?

A4. Let N be the sublattice of $\mathbb{Z}^3$ generated by the vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}-1\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\3\end{pmatrix}$, and let $M = \mathbb{Z}^3/N$.

5pt (a) Find the invariant factors of M.

5pt (b) Determine the cardinality of M.

5pt **A5.** Let G be an (additively written) abelian group defined by its generators u_1, u_2, u_3, u_4 and relations $2u_1 + 4u_2 + 10u_3 + 2u_4 = 0$ and $4u_1 = 2u_2 + 6u_4$. Represent G as a product of cyclic groups.

5pt **12.2.4.** Prove that 3×3 matrices over a field are similar iff they have the same characteristic and the same minimal polynomials.

5pt **12.2.10.** Find all similarity classes of 6×6 matrices over $\mathbb{Q}$ with minimal polynomial $(x+2)^2(x-1)$.

5pt **12.2.18.** Let V be a finite dimensional vector space over $\mathbb{Q}$ and suppose T is a nonsingular (invertible) linear transformation of V such that $T^{-1} = T^2 + T$. Prove that $\dim V$ is divisible by 3 and that all such transformations of V are similar.