Homework 8 **Math 5591H**

Due by Tuesday, March 26

Homework submitted three days after the deadline will have a 5 point penalty for each additional day of delay.

13.2.13. Suppose $F = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ where $\alpha_i^2 \in \mathbb{Q}$ for all i. Prove that $\sqrt[3]{2} \notin F$.

13.2.14. Prove that if $[F(\alpha) : F]$ is odd, then $F(\alpha) = F(\alpha^2)$.

13.2.16. Let K/F be an algebraic extension and let R be a ring with $F \subseteq R \subseteq K$. Prove that R is a field.

13.2.17. Let $f \in F[x]$ be irreducible with $\deg f = n$, and let $g \in F[x]$. Prove that every irreducible factor of $f(g(x))$ has degree divisible by n.

Let K/F be a finite extension, $\alpha \in K$, and T be the linear transformation of K as an F-vector space defined by $T(\beta) = \alpha\beta$, $\beta \in K$; then the minimal polynomial of α is the same as the minimal polynomial of T.

13.2.20. Find the minimal polynomial of $1 + \sqrt[3]{2} + \sqrt[3]{4}$ over $\mathbb{Q}$.

13.2.22. Let K_1/F and K_2/F be finite subextensions of an extension K/F. Prove that $K_1 \otimes_F K_2$ is a field iff $[K_1K_2 : F] = [K_1 : F] \cdot [K_2 : F]$. (*Hint:* Consider the natural homomorphism $K_1 \otimes_F K_2 \longrightarrow K_1K_2$.)

13.1.1. Show that $p = x^3 + 9x + 6$ is irreducible over $\mathbb{Q}$. Let θ be a root of p (in some extension of $\mathbb{Q}$); represent $(1+\theta)^{-1}$ in the form $a + b\theta + c\theta^2$ with $a, b, c \in \mathbb{Q}$.

13.1.3. Show that $p = x^3 + x + 1$ is irreducible over $\mathbb{F}_2$. Let θ be a root of p (in some extension of $\mathbb{F}_2$); compute the powers of θ in $\mathbb{F}_2(\theta)$ (in the form $a + b\theta + c\theta^2$).

13.4.2. Determine the splitting field (as a subfield of $\mathbb{C}$) and find its degree over $\mathbb{Q}$ of $f = x^4 + 2$. (*Hint:* The degree is 8.)

13.4.4. Determine the splitting field (as a subfield of $\mathbb{C}$) and find its degree over $\mathbb{Q}$ of $f = x^6 - 4$. (*Hint:* The degree is 6.)

13.4.3. Determine the splitting field (as a subfield of $\mathbb{C}$) and find its degree over $\mathbb{Q}$ of $f = x^4 + x^2 + 1$. (*Hint:* The degree is 2.)