**Midterm** **Math 5591H**

You don't have to write very detailed proofs. You may use any fact proven in class, textbook, or homework (except, of course, the very fact you are supposed to prove here).

15% **1.** Let $R$ be an integral domain, $F$ be the field of fractions of $R$, and $M$ be an $R$-module. Prove that $M/\operatorname{Tor}(M)$ is isomorphic to an $R$-submodule of an $F$-vector space.

15% **2.** Let $G = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}^2$. Recall that if $p \in \mathbb{Z}$ is prime, then $\mathbb{F}_p = \mathbb{Z}_p$ is a field.

(a) Find the dimension of the $\mathbb{F}_2$-vector space $\mathbb{Z}_2 \otimes_{\mathbb{Z}} G$.

(b) Find the dimension of the $\mathbb{F}_3$-vector space $\mathbb{Z}_3 \otimes_{\mathbb{Z}} G$.

(c) Find the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} G$.

**3.** Let $M$ and $N$ be free modules over an integral domain $R$ with $\operatorname{rank} M = \operatorname{rank} N < \infty$ and let $\varphi\colon M \longrightarrow N$ be a homomorphism.

10% (a) If $\varphi$ is surjective, prove that $\varphi$ is an isomorphism.

10% (b) If $\varphi$ is injective, prove that $N/\varphi(M)$ is a torsion module.

**4.** Let $R$ be a commutative unital ring, $I$ be an ideal in $R$, and $M$ be an $R$-module.

10% (a) Use the exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$ to prove that $(R/I) \otimes M \cong M/IM$.

10% (b) If $M$ is flat, prove that $I \otimes M \cong IM$.

15% **5.** Suppose $M$ is a free module of finite rank over a PID and $N$ is a submodule of $M$. Prove that $N$ is a free summand of $M$ (i.e. there exists a submodule $N'$ of $M$ such that $N \oplus N' = M$) iff the module $M/N$ is torsion-free.

15% **6.** Let $V$ be an $n$-dimensional vector space, let $T$ be a linear transformation $V \longrightarrow V$, and assume that the minimal polynomial $m_T$ of $T$ has degree $n$. Prove that the action of $T$ on $V$ is cyclic ($V$ is a cyclic $F[x]$-module).