Math 5591H **Some practice problems**

1. If K/F is a Galois extension of degree $n = mp^r$ where p is prime and $p \nmid m$, prove that K/F has a subextension L/F such that $[L : F] = m$, and that all such subextensions are isomorphic.

2. Let p be a prime integer, let F be a field with $\operatorname{char} F \neq p$, let $f \in F[x]$ be a separable polynomial, and assume that the splitting field K of f has degree p^r over F for some $r \in \mathbb{N}$. Prove that f is solvable in radicals. If F contains a root of unity of degree p, how many nested radicals and of what degrees would suffice to express a root of f?

3. (a) Is it true that a normal extension of a normal extension is normal? (Prove or give a counterexample.)

(b) Is it true that a separable extension of a separable extension is separable?

4. Prove that every root of unity of degree n is expressible in radicals of degrees $< n$.

5. Let K/F be a Galois extension with $\operatorname{Gal}(K/F) = G$ and let $\alpha \in K$.

(a) Prove that $K = F(\alpha)$ iff the elements $\varphi(\alpha)$, $\varphi \in G$, are all distinct.

(b) In general, prove that $[K : F(\alpha)] = |H|$ where H is the stabilizer of α in G, $H = \{\varphi \in G : \varphi(\alpha) = \alpha\}$.

6. Let K/F be a Galois extension of degree pq where $p < q$ are primes. How many subextensions and of what degrees can K/F have? (Consider two cases: where p divides $q - 1$ and where it doesn't.)

7. If $\operatorname{char} F \neq 0$, prove that an extension K/F of degree 4 can be generated by the root of an irreducible biquadratic $x^4 + ax^2 + b \in F[x]$ if and only if K contains a quadratic extension of F.

8. Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be a squarefree integer and let $a \in \mathbb{Q}$ be a nonzero rational number. Prove that the extension $\mathbb{Q}(\sqrt{a\sqrt{d}})/\mathbb{Q}$ is Galois only if $d = -1$.

9. Construct a polynomial over $\mathbb{Q}$ whose Galois group is isomorphic to $\mathbb{Z}_4$.

10. For which n is the number $\sqrt[n]{3}$ constructible?

11. Find the Galois group of $f = x^3 - 3x + 3 \in \mathbb{Q}[x]$.

12. Find the Galois group of $f = x^4 - 2$ (a) over $\mathbb{Q}$; (b) over $\mathbb{F}_3$; (c) over $\mathbb{F}_7$.

13. Find the Galois group over $\mathbb{Q}$ of the polynomials (a) $f = x^5 - 2$; (b) $f = x^9 - 2$.

14. Find the Galois group of $f = x^4 + x^3 + x^2 + x + 1$ (a) over $\mathbb{Q}$; (b) over $\mathbb{F}_2$.

15. Find the Galois group and all subfields of the splitting field of $f = x^4 + 3x^2 + 1 \in \mathbb{Q}[x]$.

16. Find the Galois group and all subfields of the splitting field of $f = x^4 + x^2 + 1 \in \mathbb{Q}[x]$.

17. Find the Galois group of $f = x^4 + 2x^2 + x + 3 \in \mathbb{Q}[x]$.

18. For prime p, prove that the Galois group of $f = x^4 + px + p \in \mathbb{Q}[x]$ is S_4 for $p \neq 3,\ 5$, D_8 for $p = 3$, and $\mathbb{Z}_4$ for $p = 5$.

19. Find the Galois group of $f = x^5 - x - 1 \in \mathbb{Q}[x]$.