Let R be a ring.

5pt **A1.** *If R is an integral domain and M is an R-module, prove that the quotient module $\widetilde{M} = M/\operatorname{Tor}(M)$ is torsion-free, that is, $\operatorname{Tor}(\widetilde{M}) = 0$.*

Solution. Let $\overline{u} \in \widetilde{M}$ and $a \in R$, $a \neq 0$, be such that $a\overline{u} = 0$ in $\widetilde{M}$. Let $\overline{u} = u + \operatorname{Tor}(M)$; then we have $au \in \operatorname{Tor}(M)$, that is there is $b \neq 0$ such that $b(au) = 0$. Since $ba \neq 0$, we obtain that $u \in \operatorname{Tor}(M)$, that is, $\overline{u} = 0$ in $\widetilde{M}$.

10pt **A2.** *Let M be a left R-module, A be an abelian group, H be the set of homomorphisms from the additive group of M to A (that is, mappings $f\colon M \longrightarrow A$ such that $f(u+v) = f(u) + f(v)$ for all $u, v \in M$). Let's define the addition on H by $(f+g)(u) = f(u) + g(u)$ and the "multiplication by scalars" by $(af)(u) = f(au)$, $f, g \in H$, $a \in R$, $u \in M$. Prove that these operations turn H into a RIGHT R-module.*

Solution. The addition is well defined on H: for any $f, g \in H$ and any $u, v \in M$, $(f+g)(u+v) = f(u+v) + g(u+v) = f(u) + f(v) + g(u) + g(v) = f(u) + g(u) + f(v) + g(v) = (f+g)(u) + (f+g)(v)$, so $f+g$ is a homomorphism $M \longrightarrow A$, that is, an element of H. And under this addition H is clearly an abelian group: for any $f, g, h \in H$, $(f+g)+h = f+(g+h)$ and $f+g = g+f$ since this is so on every element of M: for any $u \in M$, $((f+g)+h)(u) = (f+g)(u) + h(u) = f(u) + g(u) + h(u) = f(u) + g + h)(u) = (f+(g+h))(u)$ and $(f+g)(u) = f(u) + g(u) = g(u) + f(u) = (g+f)(u)$; the zero homomorphism, $0(u) = 0$ for all $u \in M$, is the neutral element of H; and for $f \in H$, $-f$ is the homomorphism $(-f)(u) = -f(u)$, $u \in M$. (Actually, for any two abelian groups B and A, $\operatorname{Hom}(B, A)$ is an abelian group under addition.)

The multiplication by scalars is also well defined: for $f \in H$ and $a \in R$, af is an element of H: for any $u, v \in M$, $(af)(u+v) = f(a(u+v)) = f(au+av) = f(au) + f(av) = (af)(u) + (af)(v)$. The distributive laws also hold since they hold elementwise: for any $a, b \in R$ and $f, g \in H$, for any $u \in M$, $a(f+g)(u) = (f+g)(au) = f(au) + g(au) = (af)(u) + (ag)(u)$ so $a(f+g) = af + ag$, and $((a+b)f)(u) = f((a+b)u) = f(au+bu) = f(au) + f(bu) = (af)(u) + (bf)(u)$ so $(a+b)f = af + bf$.

And finally, for any $a, b \in R$ and $f \in H$, for any $u \in M$, $((ab)f)(u) = f(abu) = (af)(bu) = (b(af))(u)$, so $(ab)f = b(af)$. This is the right R-module structure! We should write fa instead of af, we would then have $f(ab) = (fa)b$.

5pt **10.3.7.** *Let M be an R-module and N be its submodule. If both N and M/N are finitely generated, prove that M is finitely generated. More exactly: if N is generated by n elements and M/N is generated by m elements, prove that M is generated by $n+m$ elements.*

Solution. Let $N = R\{u_1, \ldots, u_n\}$ and $M/N = R\{\overline{v}_1, \ldots, \overline{v}_m\}$. For each $i = 1, \ldots, m$ let $v_i \in M$ be such that $\overline{v}_i = v_i + N$. I claim that $u_1, \ldots, u_n, v_1, \ldots, v_m$ generate M. Indeed, let $w \in M$ and let $\overline{w} = w + N \in M/N$. There exist $b_1, \ldots, b_m \in R$ such that $\overline{w} = b_1\overline{v}_1 + \cdots + b_m\overline{v}_m$ in M/N, so $w - (b_1v_1 + \cdots + b_mv_m) \in N$, so there are $a_1, \ldots, a_n \in R$ such that $w - (b_1v_1 + \cdots + b_mv_m) = a_1u_1 + \cdots + a_nu_n$, so $w = b_1v_1 + \cdots + b_mv_m + a_1u_1 + \cdots + a_nu_n$.

A3. *Let M be an R-module and let I, J be right ideals in R.*

5pt (a) Prove that $\operatorname{Ann}(I)$ (and $\operatorname{Ann}(J)$) is a submodule of M.

Solution. $\operatorname{Ann}(I)$ is a group under addition: if $Iu = 0$ and $Iv = 0$, then $I(u-v) \subseteq Iu - Iv = 0$. Hence, $\operatorname{Ann}(I) - \operatorname{Ann}(I) \subseteq \operatorname{Ann}(I)$.

And $\operatorname{Ann}(I)$ is invariant under multiplication by scalars: for any $c \in R$, if $Iu = 0$, then $I(cu) = (Ic)u \subseteq Iu = 0$ (since I is a right ideal and so $cI \subseteq I$). Hence, $c\operatorname{Ann}(I) \subseteq \operatorname{Ann}(I)$.

5pt (b) *Prove that* $\operatorname{Ann}(I+J) = \operatorname{Ann}(I) \cap \operatorname{Ann}(J)$.

Solution. If $u \in \operatorname{Ann}(I+J)$ then $u \in \operatorname{Ann}(I)$ and $u \in \operatorname{Ann}(J)$, so $u \in \operatorname{Ann}(I) \cap \operatorname{Ann}(J)$. Conversely, if $u \in \operatorname{Ann}(I) \cap \operatorname{Ann}(J)$, then for any $a \in I$, $b \in J$ we have $(a+b)u = au + bu = 0$, so $u \in \operatorname{Ann}(I+J)$.

5pt (c) *Prove that* $\operatorname{Ann}(I \cap J) \supseteq \operatorname{Ann}(I) + \operatorname{Ann}(J)$.

Solution. Since $\operatorname{Ann}(I), \operatorname{Ann}(J) \subseteq \operatorname{Ann}(I \cap J)$ and $\operatorname{Ann}(I \cap J)$ is a module, we have $\operatorname{Ann}(I) + \operatorname{Ann}(J) \subseteq \operatorname{Ann}(I \cap J)$.

5pt (d) *Give an example where the inclusion in (c) is strict,* $\operatorname{Ann}(I \cap J) \neq \operatorname{Ann}(I) + \operatorname{Ann}(J)$.

Solution. Take $R = F[x, y]$ where F is a field, and $M = R/(xy)$. Then M consists of polynomials of the form $a_0 + b_1x + \cdots b_nx^n + c_1y + \cdots + c_my^m$, with "multiplication by scalars" satisfying $xy = yx = 0$. Put $I = (x)$ and $J = (y)$, then $\mathrm{Ann}(I)$ consists of polynomials of the form $c_1y + \cdots + c_my^m$, $\mathrm{Ann}(J)$ consists of polynomials of the form $b_1x + \cdots + b_nx^n$, and $\mathrm{Ann}(I) + \mathrm{Ann}(J)$ consists of all polynomials in M with zero constant term. However, $I \cap J = IJ = (xy)$, and $\mathrm{Ann}(I \cap J) = M$.

Another solution. Let R be the ring of continuous functions on the interval $[0, 2]$, $R = C([0, 2])$, let $M = R$, $I = \{f : f|_{[0,1]} = 0\}$, $J = \{f : f|_{[1,2]} = 0\}$. Then $\mathrm{Ann}(I) = J$, $\mathrm{Ann}(J) = I$, and $\mathrm{Ann}(I) + \mathrm{Ann}(J) = J + I = \{f : f(1) = 0\}$; on the other hand, $I \cap J = 0$, so $\mathrm{Ann}(I \cap J) = M$.

5pt (e) *If R is commutative and unital and I, J are comaximal (that is, $I + J = (1)$), prove that* $\mathrm{Ann}(I \cap J) = \mathrm{Ann}(I) + \mathrm{Ann}(J)$.

Solution. We need to show that $\mathrm{Ann}(I \cap J) \subseteq \mathrm{Ann}(I) + \mathrm{Ann}(J)$. Let $a \in I$ and $b \in J$ be such that $a + b = 1$. Let $u \in \mathrm{Ann}(I \cap J)$; write $u = 1u = au + bu$. Then $au \in \mathrm{Ann}(J)$ since for any $c \in J$, $ca \in JI \subseteq I \cap J$ so $cau = 0$; and similarly $bu \in \mathrm{Ann}(I)$. Hence, $u \in \mathrm{Ann}(I) + \mathrm{Ann}(J)$.

5pt **10.2.9.** *Let R be a commutative unital ring and M be an R-module. Prove that* $\mathrm{Hom}_R(R, M)$ *and M are isomorphic as R-modules.*

Solution. Define a correspondence between $\mathrm{Hom}_R(R, M)$ and M by $\varphi \leftrightarrow \varphi(1)$. This is a one-to-one correspondence because every homomorphism $\varphi\colon R \longrightarrow M$ is uniquely defined by its value $\varphi(1)$ at 1: for any $a \in R$, $\varphi(a) = a\varphi(1)$, and because for every $u \in M$, the mapping defined by $\varphi(a) = au$ is a homomorphism $R \longrightarrow M$ with $\varphi(1) = u$. Also, this correspondence is a module homomorphism, since $\varphi+\psi \leftrightarrow \varphi(1)+\psi(1)$ and $a\varphi \leftrightarrow a\varphi(1)$, $a \in R$. Hence, it is an isomorphism of R-modules.

5pt **A4.** (a) *Let A be a ring and $\varphi\colon R \longrightarrow A$ be a ring homomorphism such that $\varphi(R) \subseteq Z(A)$ (where $Z(A)$ is the center of A). Prove that under the "multiplication by scalars" defined by $a\alpha = \varphi(a)\alpha$ for $a \in R$, $\alpha \in A$, A is an R algebra.*

Solution. A is an an abelian group under addition. For any $a, b \in R$ and $\alpha, \beta \in A$ we have $\varphi(a)(\varphi(b)\alpha) = \varphi(ab)\alpha$, $(\varphi(a) + \varphi(b))\alpha = \varphi(a + b)\alpha$, and $\varphi(a)(\alpha + \beta) = \varphi(a)\alpha + \varphi(a)\beta$; all together this proves that A with the scalar multiplication as defined is an R-module. Finally, since for every $a \in R$we have $\varphi(a) \in Z(A)$, we see that $\varphi(a)(\alpha\beta) = (\varphi(a)\alpha)\beta = \alpha(\varphi(a)\beta)$ for all $\alpha, \beta \in A$. Hence, A is an R-algebra.

5pt (b) *Let A be a unital R-algebra, that is, with $1 \in A$. Find a homomorphism $\varphi\colon R \longrightarrow A$ such that $a\alpha = \varphi(a)\alpha$ for all $a \in R$, $\alpha \in A$, and show that $\varphi(R) \subseteq Z(A)$.*

Solution. Define $\varphi(a) = a1 \in A$, $a \in R$. Clearly, φ is a homomorphism $R \longrightarrow A$. For any $a \in R$ and $\alpha \in A$ we have $\varphi(a)\alpha = a1\alpha = 1a\alpha 1 = 1\alpha a1 = \alpha\varphi(a)$, so $\varphi(a) \in Z(A)$, so $\varphi(R) \subseteq Z(A)$.