Solutions to Homework 12 — Math 5591H

10pt **A1.** *Let F be a field with* $\operatorname{char} F \neq 3$*, let $d \in F$ be such that $x^3 - d$ is irreducible, let $K = F(\sqrt[3]{d})$. Obtain the formula for the norm $N_{K/F}(\alpha)$ of the element $\alpha = a + b\sqrt[3]{d} + c\sqrt[3]{d^2}$, $a, b, c \in F$, of K.*

Solution. There are different ways to find the norm of α: by finding the product of all its conjugates, by computing the constant term of its minimal polynomial, ... I'll use "the determinant formula": $N_{K/\mathbb{Q}}(\alpha) = \det T_\alpha$, where T_α is the operator of multiplication by α in K. For $\alpha = a + b\sqrt[3]{d} + c\sqrt[3]{d^2}$ in the basis $\{1, \sqrt[3]{d}, \sqrt[3]{d^2}\}$ the matrix of T_α is $\begin{pmatrix} a & cd & bd \\ b & a & cd \\ c & b & a \end{pmatrix}$, and $N_{K/\mathbb{Q}}(\alpha) = \det T = a^3 + b^3 d + c^3 d^2 - 3abcd$.

5pt **A2.** (a) *Let $\alpha = \sqrt{2\sqrt{2} + \sqrt{5\sqrt{3} + 3\sqrt{5}}} + \sqrt{6}$, let $K/\mathbb{Q}$ be the Galois closure of $\mathbb{Q}(\alpha)/\mathbb{Q}$. Prove that $[K : \mathbb{Q}] = 2^n$ for some $n \in \mathbb{N}$.*

Solution. α is contained in a field E obtained from $\mathbb{Q}$ by a sequence of composites and towers of quadratic extensions, and so $E/\mathbb{Q}$ is a 2-extension. Hence, $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a 2-extension, so $K/\mathbb{Q}$ is 2-extension, so $[K : \mathbb{Q}]$ is a power of 2.

10pt (b) *Let $\alpha = \sqrt[3]{2\sqrt[3]{2} + \sqrt[3]{5\sqrt[3]{3} + 3\sqrt[3]{5}}} + \sqrt[3]{6}$, let $K/\mathbb{Q}$ be the Galois closure of $\mathbb{Q}(\alpha)/\mathbb{Q}$. Prove that $[K : \mathbb{Q}] = 2 \cdot 3^n$ for some $n \in \mathbb{N}$.*

Solution. As allowed, let's assume that K contains an element $\gamma = \sqrt[3]{a} \notin \mathbb{Q}$ with $a \in \mathbb{Q}$. The other roots of $x^3 - a$ are $\gamma\omega$ and $\gamma\omega^2$ where $\omega = e^{2\pi i/3}$, which also $\notin \mathbb{Q}$; so, $x^3 - a$ is irreducible in $\mathbb{Q}[x]$, so $\deg_{\mathbb{Q}} \gamma = 3$. Since $K/\mathbb{Q}$ is normal, K also contains $\gamma\omega$, and so, ω.

Let $L = \mathbb{Q}(\omega)$. If F is a field containing ω then for any $a \in F$ the polynomial $x^3 - a$ is either irreducible or splits completely in F; in any case, the extension $F(\sqrt[3]{a})/F$ is normal. Hence, α is contained in a field E obtained from L by a sequence of composites and towers of Galois cubic extensions. So, E/L is a 3-extension, so $L(\alpha)/L$ is a 3-extension, so K/L is a 3 extension, so $[K : L] = 3^n$ for some $n \in \mathbb{N}$. Since $[L : \mathbb{Q}] = 2$, $[K : \mathbb{Q}] = 2 \cdot 3^n$.

14.4.6. *Let p be a prime, let $K = \mathbb{F}_p(x, y)$ (the field of rational functions over $\mathbb{F}_p$ in the variables x and y), let $F = \mathbb{F}_p(x^p, y^p) \subset K$.*

5pt (a) *Prove that $\deg_F x = \deg_F y = p$ and $[K : F] = p^2$.*

Solution. x is the root of the polynomial $f(z) = z^p - x^p \in F[z]$. In $K[x]$, $f(z) = (z - x)^p$; since $m_{x,F} \mid f$, $m_{x,F}(z) = (z-x)^k$ for $k \leq p$; since $m_{x,F}$ is irreducible and $x \notin F$, it must be that $m_{x,F}(z) = (z-x)^p = f(z)$. So, $\deg_F x = p$; similarly, $\deg_F y = p$.

Same proof shows that $\deg_{F(y)} x = p$, so $[K : F] = [F(x, y) : F(y)] \cdot [F(y) : F] = p^2$.

5pt (b) *Prove that the extension K/F is normal.*

Solution. K is generated by x and y whose minimal polynomials over F split completely in K.

5pt (c) *Prove that K is not a simple extension of F: there is no $\alpha \in K$ such that $K = F(\alpha)$.*

Solution. For any $h \in K = \mathbb{F}_p(x, y)$ we have $h(x, y)^p = h(x^p, y^p) \in F$, so $\deg_F h \leq p$ and so. $F(\alpha) \neq K$.

5pt (d) *Explain why the result of (c) doesn't contradict "the theorem on the primitive element".*

Solution. There is no contradiction because the extension K/F is not separable.

5pt **A3.** *Let θ be such that $\cos\theta = 5/7$; prove that the angle $\theta/5$ is not constructible with ruler and compass.*

Solution. Let $\alpha = \cos(\theta/5)$; then $16\alpha^5 - 20\alpha^3 + 5\alpha = \cos\theta = 5/7$, so α is a root of $f = 7 \cdot 16x^5 - 7 \cdot 20x^3 + 7 \cdot 5x - 5$. f is irreducible in $\mathbb{Q}[x]$ by Eisenstein and Gauss, so α has degree 5 over $\mathbb{Q}$, is not an element of a 2-extension of $\mathbb{Q}$, and thus is not constructible.

5pt **13.3.1,4.** *Prove that it is impossible to "construct with ruler and compass" the regular 7- and 9-gons.*

Solution. 7 is a prime but not a Fermat's prime ($7 - 1$ is not a power of 2). $9 = 3^2$ is not a product of distinct Fermat's primes.

10pt **A4.** *Prove that there are no nontrivial automorphisms of* $\mathbb{R}$.

Solution. Any automorphism of any field F always fixes the prime subfield of F: $1 \mapsto 1$, so $2 \mapsto 2$, so $n \mapsto n$ for any $n \in Z$, where Z is the additive subgroup of F generated by 1, so $1/n \mapsto 1/n$ and $m/n \mapsto m/n$ for any $m, n \in Z$ with $n \neq 0$. In particular, for any $\varphi \in \mathrm{Aut}(\mathbb{R})$ and $q \in \mathbb{Q}$, $\varphi(q) = q$.

Let $\varphi \in \mathrm{Aut}(\mathbb{R})$. For any $a > 0$ we have $\varphi(a) = \varphi\big(\sqrt{a}^2\big) = \varphi\big(\sqrt{a}\big)^2 > 0$. Thus for any $a, b \in \mathbb{R}$ with $a < b$, $\varphi(b) - \varphi(a) = \varphi(b - a) > 0$, so $\varphi(a) < \varphi(b)$.

Now, let $a \in \mathbb{R}$ and assume that $\varphi(a) \neq a$, say, $\varphi(a) < a$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ there is $q \in \mathbb{Q}$ such that $\varphi(a) < q < a$. But then $q < a$ and $\varphi(q) = q > \varphi(a)$, contradiction.

(In contrast, the group $\mathrm{Aut}(\mathbb{C}) = \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ is huge; but to prove this we need some theory of transcendental extensions.)