**Solutions to Homework 2** **Math 5591H**

**10.5.1(d,e).** *Suppose that*

$$\begin{array}{ccccc} A & \xrightarrow{\psi} & B & \xrightarrow{\varphi} & C \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow \\ A' & \xrightarrow{\psi'} & B' & \xrightarrow{\varphi'} & C' \end{array}$$

*is a commutative diagram of module homomorphisms with exact rows.*

10pt (d) *If $\beta$ is injective and $\alpha$ and $\varphi$ are surjective, prove that $\gamma$ is injective.*

*Solution.* Let $c \in C$ be such that $\gamma(c) = 0$. Since $\varphi$ is surjective, there exists $b \in B$ such that $\varphi(b) = c$. Put $b' = \beta(b)$. Then $\varphi'(b') = \gamma(\varphi(b)) = \gamma(c) = 0$. Since the second row is exact, there exists $a' \in A'$ such that $\psi'(a') = b'$. Since $\alpha$ is surjective, there exists $a \in A$ such that $\alpha(a) = a'$. Then $\beta(\psi(a)) = \psi'(\alpha(a)) = \psi'(a') = b' = \beta(b)$. Since $\beta$ is injective, $b = \psi(a)$. Since the first row is exact, $c = \varphi(\psi(a)) = 0$. Hence, $\gamma$ is injective.

10pt (e) *If $\beta$ is surjective and $\gamma$ and $\psi'$ are injective, prove that $\alpha$ is surjective.*

*Solution.* Let $a' \in A'$; we need to find $a \in A$ such that $\alpha(a) = a'$. Since $\beta$ is surjective, there exists $b \in B$ such that $\beta(b) = \psi'(a')$. Since the second row is exact, $\varphi'(\psi'(a')) = 0$, so $\gamma(\varphi(b)) = \varphi'(\beta(b)) = \varphi'(\psi'(a')) = 0$. Since $\gamma$ is injective, $\varphi(b) = 0$. Since the first row is exact, $b = \psi(a)$ for some $a \in A$. Then $\psi'(\alpha(a)) = \beta(\psi(a)) = \beta(b) = \psi'(a')$; since $\psi'$ is injective, $\alpha(a) = a'$.

**A1.** *Let*

$$\begin{array}{ccccccccc} 0 \longrightarrow & A & \xrightarrow{\psi} & B & \xrightarrow{\varphi} & C & \longrightarrow 0 \\ & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & \\ 0 \longrightarrow & A' & \xrightarrow{\psi'} & B' & \xrightarrow{\varphi'} & C' & \longrightarrow 0 \end{array}$$

*be a commutative diagram of module homomorphisms with exact rows. Define a mapping $\delta\colon \ker\gamma \longrightarrow \operatorname{coker}\alpha$ in the follwing way: for $c \in \ker(\gamma)$ choose $b \in \varphi^{-1}(c)$ and put $b' = \beta(b)$; then $\varphi'(b') = \gamma(c) = 0$, so $b' = \psi'(a')$ for some $a' \in A'$. Now put $\delta(c) = a' \bmod \alpha(A)$.*

5pt (a) *Prove that $\delta$ is well defined (i.e. doesn't depend on the choice of $b$) and is a homomorphism.*

*Solution.* To show that $\delta$ is well defined, let $b_1, b_2 \in B$ be such that $\varphi(b_1) = \varphi(b_2) = c$. Then $\beta_1 - \beta_2 \in \ker(\varphi) = \operatorname{Ran}(\psi)$, so $\beta_1 - \beta_2 = \psi(d)$ for some $d \in A$. Let $b'_1 = \beta(\beta_1)$ and $b'_2 = \beta(\beta_2)$ and $a'_1, a'_2 \in A'$ be such that $b'_1 = \psi'(a'_1)$, $b'_2 = \psi'(a'_2)$. Then $\psi'(a'_1 - a'_2) = \beta(b_1 - b_2) = \beta(\psi(d)) = \psi'(\alpha(d))$; since $\psi'$ is injective this implies that $a'_1 - a'_2 = \alpha(d)$, that is, $a'_1 = a'_2 \bmod \alpha(A)$.

To prove that $\delta$ is a homomorphism, let $c_1, c_2 \in \ker(\gamma)$ and let $b_1, b_2 \in B$ and $a'_1, a'_2 \in A'$ be such that $\varphi(b_1) = c_1$, $\varphi(b_2) = c_2$, $\psi'(a'_1) = \beta(b_1)$, and $\psi'(a'_2) = \beta(b_2)$, so that $\delta(c_1) = a'_1$ and $\delta(c_2) = a'_2$. then $\varphi(b_1 + b_2) = c_1 + c_2$ and $\psi'(a'_1 + a'_2) = \psi'(a_1) + \psi'(a_2) = \beta(b_1) + \beta(b_2) = \beta(b_1 + b_2)$, so $\delta(c_1 + c_2) = a'_1 + a'_2$. Also for any $r$ is a scalar, then $\varphi(rb_1) = rc_1$ and $\psi'(ra'_1) = r\psi'(a_1) = r\beta(b_1) = \beta(rb_1)$, so $\delta(rc_1) = ra'_1$.

10pt (b) *Prove that $\ker(\delta) = \varphi(\ker(\beta))$.*

*Solution.* Let $c \in \ker(\gamma)$, let $b$, $b'$ and $a' = \delta(c)$ be as in the construction of $\delta$.

If $c = \varphi(\ker(\beta))$, $c = \varphi(b)$ with $b' = \beta(b) = 0$, then $a' = 0$, so $\delta(c) = a' \bmod \alpha(A) = 0$, so $c \in \ker(\delta)$.

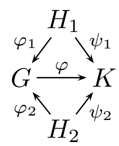
Now assume that $c \in \ker(\delta)$. Then $a' \in \alpha(A)$, that is, $a' = \alpha(a)$ for some $a \in A$. Define $d = \psi(a)$; then $\beta(b - d) = b' - \beta(\psi(a)) = b' - \psi'(\alpha(a)) = b' - \psi'(a) = 0$, so $b - d \in \ker(\beta)$. On the other hand, $\varphi(b - d) = \varphi(b) - \varphi(\psi(a)) = \varphi(b) = c$. Hence, $c \in \varphi(\ker(\beta))$.

5pt **A2.** *Consider the category where objects are integers and a morphism (an arrow) from object $n$ to object $m$ exists, and is unique, iff $n \,|\, m$. (Notice that in this category objects are not assumed to be sets and morphisms are not mappings!) The composition of two morphisms exists and is defined uniquely, since if $n \,|\, m$ and $m \,|\, k$ then $n \,|\, k$. What objects in this category are isomorphic? Are there universal repelling and/or attracting objects?*

*Solution.* Distinct $n, m \in \mathbb{Z}$ are isomorphic iff both $n \,|\, m$ and $m \,|\, n$, which is true iff $m = -n$.

The universal repelling object is 1 (and $-1$), since 1 divides every $n \in \mathbb{Z}$ and so, the (unique) morphism $1 \longrightarrow n$ exists. The universal attracting object is 0, since 0 is divisible by every integer.

10pt **A3.** *Let $H_1$ and $H_2$ be two groups. Consider the category of groups $G$ with homomorphisms $H_1, H_2 \longrightarrow G$: the objects are triplets $(G, \varphi_1, \varphi_2)$ where $G$ is a group and $\varphi_i : H_i \longrightarrow G$, $i = 1, 2$, are group homomorphisms; the morphisms $(G, \varphi_1, \varphi_2) \longrightarrow (K, \psi_1, \psi_2)$ are homomorphisms $\varphi : G \longrightarrow K$ for which the diagram*




*is commutative. Prove that in this category the universal repelling object is the free product $H_1 * H_2$ of $H_1$ and $H_2$, defined in the following way. The elements of $H_1 * H_2$ are "alternating" words of the form $a_1 \cdots a_k$ where for all $i$, $a_i \in H_1 \setminus \{1\}$ or $H_2 \setminus \{1\}$ with $a_{i+1} \in H_2$ if $a_i \in H_1$ and $a_{i+1} \in H_1$ if $a_i \in H_2$. The operation in $H_1 * H_2$ is concatenation of such words with natural reduction: if a subword $bc$ with both $b, c \in H_1$ or $b, c \in H_2$ occurs, it is replaced by the element $bc$ of this group, and in the case $bc = 1$ it is removed and the process of reduction continues.*

*Solution.* I will assume that $H_1 \cap H_2 = \emptyset$. (If not, I replace $H_1$ and $H_2$ by their disjoint copies.) Given homomorphisms $\varphi_1 : H_1 \longrightarrow G$ and $\varphi_2 : H_2 \longrightarrow G$, we define a homomorphism $\varphi : H_1 * H_2 \longrightarrow G$ in the following way. First, for any word $a_1 \cdots a_k$ in the alphabet $H_1 \cup H_2$ we put $\varphi(a_1 \cdots a_k) = \prod_{i=1}^{k} \tau_i(a_i)$, where for every $i$, $\tau_i = \varphi_1$ is $a_i \in H_1$ and $\tau_i = \varphi_2$ is $a_i \in H_2$. Then for any two such words $w_1$ and $w_2$ we clearly have $\varphi(w_1 w_2) = \varphi(w_1)\varphi(w_2)$. $H_1 * H_2$ is the set of reduced words, with the operation of "concatenation and reduction" (as in the definition of $H_1 * H_2$); if a word $w'$ is obtained from a word $w$ by reduction then $\varphi(w') = \varphi(w)$, since for any two elements $b$ and $c$ belonging to one of $H_1$ or $H_2$ we have $\varphi(bc) = \varphi(b)\varphi(c)$, and since $\varphi(1_{H_1}) = \varphi(1_{H_2}) = 1$. Hence, $\varphi$ restricted on $H_1 * H_2$ is a homomorphism to $G$. $H_1$ and $H_2$ are naturally subgroups of $H_1 * H_2$, and we have $\varphi|_{H_i} = \varphi_i$, $i = 1, 2$, so $\varphi$ is a morphism in our category. Such morphism is unique since $H_1$ and $H_2$ generate $H_1 * H_2$, and so, $\varphi$ is uniquely defined by $\varphi|_{H_1}$ and $\varphi|_{H_2}$.

**10.3.12.** *Let $R$ be a commutative ring and let $A$, $B$, and $M$ be $R$-modules. Prove the following isomorphisms of $R$-modules:*

5pt (a) $\mathrm{Hom}_R(A \oplus B, M) \cong \mathrm{Hom}_R(A, M) \oplus \mathrm{Hom}_R(B, M)$.

*Solution.* Define a mapping $\Phi : \mathrm{Hom}_R(A \oplus B, M) \longrightarrow \mathrm{Hom}_R(A, M) \oplus \mathrm{Hom}_R(B, M)$ by $\Phi(\varphi) = (\varphi_1, \varphi_2)$ where $\varphi_1(u) = \varphi(u, 0)$, $u \in A$, and $\varphi_2(v) = \varphi(0, v)$, $v \in B$. The inverse mapping $\Psi : \mathrm{Hom}_R(A, M) \oplus \mathrm{Hom}_R(B, M) \longrightarrow \mathrm{Hom}_R(A \oplus B, M)$ is given by $(\Psi(\varphi_1, \varphi_2))(u, v) = \varphi_1(u) + \varphi_2(v)$. Hence, $\Phi$ is a bijection, and clearly $\Phi$ is a homomorphism of $R$-modules. (A sum of homomorphisms goes to a sum and a multiple goes to a multiple.) So, $\Phi$ is an isomorphism.

5pt (b) $\mathrm{Hom}_R(M, A \oplus B) \cong \mathrm{Hom}_R(M, A) \oplus \mathrm{Hom}_R(M, B)$.

*Solution.* Let $\pi_1$ and $\pi_2$ be the projections of $A \oplus B$ onto $A$ and $B$ respectively. Define a mapping $\Phi : \mathrm{Hom}_R(M, A \oplus B) \longrightarrow \mathrm{Hom}_R(M, A) \oplus \mathrm{Hom}_R(M, B)$ by $\Phi(\varphi) = (\varphi_1, \varphi_2)$ where $\varphi_1(u) = \pi_1(\varphi(u))$ and $\varphi_2(u) = \pi_2(\varphi(u))$. The inverse mapping $\Psi : \mathrm{Hom}_R(M, A) \oplus \mathrm{Hom}_R(M, B) \longrightarrow \mathrm{Hom}_R(M, A \oplus B)$ is $(\Psi(\varphi_1, \varphi_2))(u) = (\varphi_1(u), \varphi_2(u))$. Hence, $\Phi$ is a bijection, and clearly $\Phi$ is a homomorphism of $R$-modules. (A sum of homomorphisms goes to a sum and a multiple goes to a multiple.) So, $\Phi$ is an isomorphism.

5pt **10.3.15.** *If $e$ is a central idempotent in a unital ring $R$ and $M$ is an $R$-module, prove that $eM = \mathrm{Ann}(1 - e)$, $(1 - e)M = \mathrm{Ann}(e)$, and $M = eM \oplus (1 - e)M$.*

*Solution.* If $e$ is an idempotent, then $1 - e$ is also an idempotent: $(1 - e)^2 = 1 - 2e + e^2 = 1 - 2e + e = 1 - e$.

If $u = ev$ for some $v$, then $(1 - e)u = ev - e^2 v = 0$; conversely, if $(1 - e)u = 0$, then $u = eu$; so, $eM = \mathrm{Ann}(1 - e)$. Since $e$ and $1 - e$ are switchable, also $(1 - e)M = \mathrm{Ann}(e)$.

Since $e$ (and so $(1 - e)$) is a central element of $R$, $eM$ and $(1 - e)M$ are submodules of $M$.

For any $u \in M$ we have $u = eu + (1 - e)u$ with $eu \in eM$ and $(1 - e)u \in (1 - e)M$, so $M = eM + (1 - e)M$. Also, $eM \cap (1 - e)M = 0$: indeed, if $eu = (1 - e)v$, then $eu = e^2 u = e(1 - e)v = e - e^2 v = 0$. Hence, $M = eM \oplus (1 - e)M$.

5pt **11.2.11(a,b).** *Let $R$ be a unital ring and let $\varphi$ be an endomorphism of an $R$-module $M$ such that $\varphi^2 = \varphi$. Prove that $M = \varphi(M) \oplus \ker\varphi$.*

*Solution.* Define the multiplication by $\varphi$ by $\varphi u = \varphi(u)$, $u \in M$. Since for every $a \in R$ we have $(\varphi a)(u) = \varphi(au) = a\varphi(u) = (a\varphi)(u)$, $u \in M$. $M$ gets a structure of an $R[\varphi]$-module, where $(a_n\varphi^n + \cdots + a_1\varphi + a_0)u = a_n\varphi^n(u) + \cdots + a_1\varphi(u) + a_0 u$, $u \in M$. Since $\varphi^2 = \varphi$, $\varphi$ is a central idempotent in $R[\varphi]$. By 10.3.15, $M = \varphi(M) \oplus (1-\varphi)(M)$ and $(1-\varphi)(M) = \mathrm{Ann}(\varphi) = \ker\varphi$.

*Another solution.* We need to prove that $\varphi(M) \cap \ker\varphi = 0$ and $M = \varphi(M) + \ker\varphi$.

If $u \in \varphi(M) \cap \ker\varphi$, then $u = \varphi(v)$ for some $v \in M$ and $\varphi(u) = 0$, so $0 = \varphi(u) = \varphi^2(v) = \varphi(v) = u$.

Let $u \in M$; then $u = \varphi(u) + (u - \varphi(u))$. We have $\varphi(u) \in \varphi(M)$, and $\varphi(u - \varphi(u)) = \varphi(u) - \varphi^2(u) = \varphi(u) - \varphi(u) = 0$, so $u - \varphi(u) \in \ker\varphi$; hence, $M = \varphi(M) + \ker\varphi$.