In all problems, all modules are assumed to be over a commutative unital ring R.

5pt **10.4.14,15.** (a) *Prove that for any modules M_1, M_2, and N, $(M_1 \oplus M_2) \otimes N \cong (M_1 \otimes N) \oplus (M_2 \otimes N)$, with $(u_1, u_2) \otimes v \leftrightarrow \big((u_1 \otimes v), (u_2 \otimes v)\big)$.*

Solution. Define $\beta \colon (M_1 \oplus M_2) \times N \longrightarrow (M_1 \otimes N) \oplus (M_2 \otimes N)$ by $\beta((u_1, u_2), v) = \big(u_1 \otimes v, u_2 \otimes v\big)$, $u_1 \in M_1$, $u_2 \in M_2$, $v \in N$. It is easy to check that β is bilinear: for any $u_1, u_1' \in M_1$, $u_2, u_2' \in M_2$, and $v \in N$, $\beta((u_1, u_2) + (u_1', u_{2'}), v) = \big((u_1 + u_1') \otimes v, (u_2 + u_2') \otimes v\big) = \big(u_1 \otimes v + u_1' \otimes v, u_2 \otimes v + u_2' \otimes v\big) = (u_1 \otimes v, u_2 \otimes v) + (u_1' \otimes v, u_2' \otimes v)$, etc. Hence, β induces a homomorphism $\varphi \colon (M_1 \oplus M_2) \otimes N \longrightarrow (M_1 \otimes N) \oplus (M_2 \otimes N)$ with $\varphi((u_1, u_2) \otimes v) = \big(u_1 \otimes v, u_2 \otimes v\big)$ for all $u_1 \in M_1$, $u_2 \in M_2$, $v \in N$. In particular, $\varphi((u_1, 0) \otimes v) = (u_1 \otimes v, 0)$ and $\varphi((0, u_2) \otimes v) = (0, u_2 \otimes v)$ for all $u_1 \in M_1$, $u_2 \in M_2$, $v \in N$.

On the other hand, we have homomorphisms $\psi_1 \colon M_1 \otimes N \longrightarrow (M_1 \oplus M_2) \otimes N$ and $\psi_2 \colon M_2 \otimes N \longrightarrow (M_1 \oplus M_2) \otimes N$ with $\psi_1(u_1 \otimes v) = (u_1, 0) \otimes v$ and $\psi_2(u_2 \otimes v) = (0, u_2) \otimes v$, for all $u_1 \in M_1$, $u_2 \in M_2$, $v \in N$. (Indeed, the mappings $M_1 \times N \longrightarrow (M_1 \oplus M_2) \otimes N$ and $M_2 \times N \longrightarrow (M_1 \oplus M_2) \otimes N$ given by $(u_1 \otimes v) = (u_1, 0) \otimes v$ and $(u_2 \otimes v) = (0, u_1) \otimes v$ respectively are bilinear.) Hence (by the universal property of the direct sum) there is a homomorphism $\psi \colon (M_1 \otimes N) \oplus (M_2 \otimes N) \longrightarrow (M_1 \oplus M_2) \otimes N$ such that $\psi(u_1 \otimes v, 0) = (u_1, 0) \otimes v$ and $\psi(0, u_2 \otimes v) = (0, u_2) \otimes v$ for all $u_1 \in M_1$, $u_2 \in M_2$, $v \in N$. Thus, φ and ψ are inverses of each other on the generators of $(M_1 \oplus M_2) \otimes N$ and $(M_1 \otimes N) \oplus (M_2 \otimes N)$, and so, φ is an isomorphism.

5pt (b) *Prove that for any family $(M_\alpha)_{\alpha \in \Lambda}$ of modules and any module N, $\big(\bigoplus_{\alpha \in \Lambda} M_\alpha\big) \otimes N \cong \bigoplus_{\alpha \in \Lambda}(M_\alpha \otimes N)$.*

Solution. Since Λ can be infinite, we cannot use (a) and induction (at least the conventional one). But we can simply mimic the proof of (a): Define $\beta \colon \big(\bigoplus_{\alpha \in \Lambda} M_\alpha\big) \times N \longrightarrow \bigoplus_{\alpha \in \Lambda} M_\alpha \otimes N$ by $\beta((u_\alpha)_{\alpha \in \Lambda}, v) = (u_\alpha \otimes v)_{\alpha \in \Lambda}$. It is easy to see that β is bilinear, hence it induces a homomorphism $\varphi \colon \big(\bigoplus_{\alpha \in \Lambda} M_\alpha\big) \otimes N \longrightarrow \bigoplus_{\alpha \in \Lambda} M_\alpha \otimes N$ with $\varphi((u_\alpha)_{\alpha \in \Lambda} \otimes v) = (u_\alpha \otimes v)_{\alpha \in \Lambda}$.

On the other hand, for every $\alpha \in \Lambda$, since the mapping $M_\alpha \otimes N \longrightarrow \big(\bigoplus_{\alpha \in \Lambda} M_\alpha\big) \otimes N$ given by $(u \otimes v) = (u_\delta)_{\delta \in \Lambda} \otimes v$ where $u_\delta = u$ if $\delta = \alpha$ and 0 otherwise, is bilinear, we have the homomorphism $\psi_\alpha \colon M_\alpha \otimes N \longrightarrow \big(\bigoplus_{\alpha \in \Lambda} M_\alpha\big) \otimes N$ defined by $\psi_\alpha(u \otimes v) = (u_\delta)_{\delta \in \Lambda} \otimes v$ with $u_\delta = u$ if $\delta = \alpha$ and 0 otherwise. Hence (by the universal property of the direct sum) there is a homomorphism $\psi \colon \bigoplus_{\alpha \in \Lambda}(M_\alpha \otimes N) \longrightarrow \big(\bigoplus_{\alpha \in \Lambda} M_\alpha\big) \otimes N$ satisfying, in particular, $\psi\big((u_\alpha \otimes v)_{\alpha \in \Lambda}\big) = \sum_{\alpha \in \Lambda}^{\text{fin}} \varphi_\alpha(u_\alpha \otimes v) = (u_\alpha)_{\alpha \in \Lambda} \otimes v$. Thus, φ and ψ are inverses of each other on the generators, so are the inverses of each other, and so, φ is an isomorphism.

10pt (c) *Give an example where $\big(\prod_{\alpha \in \Lambda} M_\alpha\big) \otimes N \not\cong \prod_{\alpha \in \Lambda}(M_\alpha \otimes N)$.*

Solution. For every $n \in \mathbb{N}$ let M_n be the $\mathbb{Z}$-module $\mathbb{Z}_n$. Since M_n is a torsion module, $M_n \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, so $\prod_{n=1}^{\infty} M_n \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. The product $M = \prod_{n=1}^{\infty} M_n$ is not a torsion module: for $u = (1, 1, 1, \ldots)$, for any nonzero $n \in \mathbb{Z}$, $n \neq 0$ in M_{n+1} (or M_{-n+1} if $n < 0$), so $nu \neq 0$. Hence, $M \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$.

10.4.10. *Let $N \cong R^n$ be a free module with basis $\{e_1, \ldots, e_n\}$.*

5pt (a) *Let M be a nonzero R-module. Show that every element of $M \otimes N$ can be written uniquely in the form $\sum_{i=1}^{n} m_i \otimes e_i$ where $m_i \in M$. Deduce that if $\sum_{i=1}^{n} m_i \otimes e_i = 0$ then $m_i = 0$ for all i.*

Solution. As we know, there is an isomorphism $M \otimes N \cong M \otimes R^n \cong (M \otimes R)^n \cong M^n$ with $m \otimes (a_1 e_1 + \cdots + a_n e_n) \leftrightarrow (a_1 m, \ldots, a_n m)$, $m \in M$, $a_1, \ldots, a_n \in R$. Under this isomorphism, every tensor $\sum_{i=1}^{n} m_i \otimes e_i$ corresponds to the element $(m_1, \ldots, m_n)$ of M^n, whose "coordinates" $m_1, \ldots, m_n$ are uniquely defined.

5pt (b) *If $\{e_1, \ldots, e_n\}$ isn't a basis but only a maximal linearly independent subset of N, show that the second assertion in (a) needs not be true.*

Solution. Take $R = \mathbb{Z}$, $N = \mathbb{Z}$, $e_1 = 2 \in N$, $M = \mathbb{Z}_2$, and $m_1 = 1 \in M$ so that $w = m_1 \otimes e_1 = 1 \otimes 2$; then $w = 2 \otimes 1 = 0 = m_1 \otimes 0$.

5pt **10.4.11.** *Let $\{e_1, e_2\}$ be a basis in $V = \mathbb{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ is not (cannot be written as) a simple tensor in $V \otimes_{\mathbb{R}} V$.*

Solution. Every simple tensor in $V \otimes V$ has form $u \otimes v$ for some $u = a_1 e_1 + a_2 e_2 \in V$ and $v = b_1 e_1 + b_2 e_2 \in V$, and in the basis $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$ is written as

$$a_1 b_1 e_1 \otimes e_1 + a_1 b_2 e_1 \otimes e_2 + a_2 b_1 e_2 \otimes e_1 + a_2 b_2 e_2 \otimes e_2.$$

Thus, the matrix of the coordinates of $u \otimes v$ in this basis is $\begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix}$. This matrix is degenerate (has zero determinant), whereas the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of the coordinates of $e_1 \otimes e_2 + e_2 \otimes e_1$ is nondegenerate. Hence, this tensor is not simple.

Or: If $a_1 b_1 e_1 \otimes e_1 + a_1 b_2 e_1 \otimes e_2 + a_2 b_1 e_2 \otimes e_1 + a_2 b_2 e_2 \otimes e_2 = e_1 \otimes e_2 + e_2 \otimes e_1$, then $a_1 b_1 = a_2 b_2 = 0$ and $a_1 b_2 = a_2 b_1 = 1$, which is impossible.

10.4.16. *Let I, J be ideals in R.*

5pt (a) *Prove that every element of $R/I \otimes R/J$ can be written as a simple tensor of the form $(1 \bmod I) \otimes (r \bmod J)$.*

Solution. For any $a_i, b_i, c_i \in R$ we have

$$\sum_{i=1}^{n} a_i \big((b_i \bmod I) \otimes (c_i \bmod J) \big) = \sum_{i=1}^{n} a_i b_i c_i \big((1 \bmod I) \otimes (1 \bmod J) \big) = r \big((1 \bmod I) \otimes (1 \bmod J) \big)$$
$$= (1 \bmod I) \otimes (r \bmod J),$$

where $r = \sum_{i=1}^{n} a_i b_i c_i$.

5pt (b) *Prove that there is an isomorphism $R/I \otimes R/J \cong R/(I+J)$ with $(r \bmod I) \otimes (r' \bmod J) \mapsto rr' \bmod (I+J)$, $r, r' \in R$.*

Solution. The mapping $R/I \times R/J \longrightarrow R/(I+J)$, $(r \bmod I) \otimes (r' \bmod J) \mapsto rr' \bmod (I+J)$, is well defined (since for any $a, b \in R$, $c \in I$, and $d \in J$ we have $(a+c)(b+d) = ab \bmod (I+J)$) and bilinear, thus it induces a homomorphism $\varphi \colon R/I \otimes R/J \cong R/(I+J)$ with $\varphi \colon (r \bmod I) \otimes (r' \bmod J) \mapsto rr' \bmod (I+J)$ for all $r, r' \in R$.

Define a homomorphism $\psi \colon R \longrightarrow R/I \otimes R/J$ by $\psi(r) = (1 \bmod I) \otimes (r \bmod J)$. Then $I, J \subseteq \ker(\psi)$, so $I + J \subseteq \ker(\psi)$: indeed, if $r \in J$ then $\psi(r) = (1 \bmod I) \otimes (r \bmod J) = (1 \bmod I) \otimes 0 = 0$, and if $r \in I$ then $\psi(r) = (1 \bmod I) \otimes (r \bmod J) = (r \bmod I) \otimes (1 \bmod J) = 0$. So, ψ factorizes to a homomorphism $R/(I+J) \longrightarrow R/I \otimes R/J$. I now claim that $\psi = \varphi^{-1}$, this will prove that φ is an isomorphism. Indeed,

$$\varphi \big(\psi(r \bmod (I+J)) \big) = \varphi \big((1 \bmod I) \otimes (r \bmod J) \big) = r \bmod (I+J).$$

Now let w be a simple tensor from $R/I \otimes R/J$, $w = (r \bmod I) \otimes (r' \bmod J)$. Then

$$\psi(\varphi(w)) = \psi(rr' \bmod (I+J)) = (1 \bmod I) \otimes (rr' \bmod J) = r(1 \bmod I) \otimes (r' \bmod J) = (r \bmod I) \otimes (r' \bmod J) = w.$$

Another solution. We have $(R/I) \otimes (R/J) \cong (R/J)/(I(R/J))$. The preimage (the inverse image) in R of the submodule $I(R/J)$ of R/J is $I + J$, so by the 3rd isomorphism theorem, $(R/J)/(I(R/J)) \cong R/(I+J)$.

10.4.21. *Let I and J be ideals in R.*

5pt (a) *Show that there exists a surjective homomorphism $\varphi \colon I \otimes_R J \longrightarrow IJ$ with $\varphi(i \otimes j) = ij$ for all $i \in I$, $j \in J$.*

Solution. The mapping $(i, j) \mapsto ij$ is bilinear on $I \times J$, thus extends to a homomorphism $\varphi \colon I \otimes J \longrightarrow IJ$. To show that φ is surjective, let $a \in IJ$; then $a = \sum_{k=1}^{n} i_k j_k$ for some $i_1, \ldots, i_n \in I$, $j_1, \ldots, j_n \in J$. Put $u = \sum_{k=1}^{n} i_k \otimes j_k \in I \otimes J$, then $a = \varphi(u)$.

10pt (b) *Give an example to show that φ may not be injective.*

Solution. Take $R = \mathbb{Z}_4$ and $I = J = (2) = \{0, 2\}$; then $IJ = 0$. I claim that $I \otimes J \neq 0$, so φ cannot be injective. Indeed, I and J are, actually, $\mathbb{Z}_4/(2) = \mathbb{Z}_2$-modules, both isomorphic to $\mathbb{Z}_2$, so $I \otimes_{\mathbb{Z}_4} J = I \otimes_{\mathbb{Z}_2} J \cong \mathbb{Z}_2$.

Or: Define a mapping $I \times J \longrightarrow \mathbb{Z}_2$ by $(0,0), (0,2), (2,0) \mapsto 0$ and $(2,2) \mapsto 1$; it is easy to check that this mapping is bilinear and so, defines a homomorphism onto $\mathbb{Z}_2$.

10.4.25. *Let S be an R-algebra; prove that $S \otimes_R R[x] \cong S[x]$ as R-algebras.*

Solution. $R[x]$ is a free R-module with basis $\{1, x, x^2, \ldots\}$, thus $S \otimes_R R[x]$ is the direct sum $(S \otimes 1) \oplus (S \otimes x) \oplus (S \otimes x^2) \oplus \cdots$. Hence, the natural R-module isomorphism $S \otimes_R R[x] \longrightarrow S[x]$, which maps $\alpha \otimes x^n \mapsto \alpha x^n$, $\alpha \in S$, $n \geq 0$, is well defined. This isomorphism is multiplicative: for any $\alpha_1, \alpha_2 \in S$ and $n_1, n_2 \geq 0$,

$$(\alpha_1 \otimes x^{n_1})(\alpha_2 \otimes x^{n_2}) = (\alpha_1 \alpha_2) \otimes x^{n_1 + n_2} \mapsto \alpha_1 \alpha_2 x^{n_1 + n_2} = (\alpha_1 x^{n_1})(\alpha_2 x^{n_2}),$$

so is a ring isomorphism.

A1. *Prove that $R[x] \otimes_R R[y] \cong R[x, y]$ as R-algebras.*

Solution. $R[x]$ is a free R-module with the basis $\{1, x, x^2, \ldots\}$, $R[y]$ is a free R-module with the basis $\{1, y, y^2, \ldots\}$, so $R[x] \otimes R[y]$ is a free R-module with the basis $\{x^n \otimes y^m, \ n, m = 0, 1, \ldots\}$. $R[x, y]$ is also a free R-module, with the basis $\{x^n y^m, \ n, m = 0, 1, \ldots\}$. We therefore have an R-module isomomorphism $R[x] \otimes_R R[y] \longrightarrow R[x, y]$ defined by $x^n \otimes y^m \mapsto x^n y^m$. This isomorpshism is multiplicative: $(x^{n_1} \otimes y^{m_1})(x^{n_2} \otimes y^{m_2}) = x^{n_1 + n_1} \otimes y^{m_1 + m_2} \mapsto x^{n_1 + n_1} y^{m_1 + m_2} = x^{n_1} \otimes y^{m_1} x^{n_2} \otimes y^{m_2}$, so is a ring isomorphism.

Another solution. By problem 10.4.25, $R[x] \otimes_R R[y] \cong R[x][y] = R[x, y]$.