

In all problems, all modules are assumed to be over a commutative unital ring R .

A1. The following examples show that the tensor product of torsion-free modules may have a torsion: Let $R = \mathbb{Z}[x, y]$, let I be the ideal (x, y) in R , considered as an R -module, and let $w = x \otimes y - y \otimes x \in I \otimes I$.

5pt (a) Prove that w is a torsion element of $I \otimes I$.

Solution. $xw = x^2 \otimes y - (xy) \otimes x = x \otimes (xy) - (xy) \otimes x = x \otimes (yx) - (xy) \otimes x = (yx) \otimes x - (xy) \otimes x = 0$.

5pt (b) Prove that $w \neq 0$.

Solution. Consider the mapping $\beta: I \times I \rightarrow \mathbb{Z}$ defined by $\beta(p, q) = a_1b_2$ for all polynomials $p = a_1x + a_2y + a_3xy + \dots$ and $q = b_1x + b_2y + b_3xy + \dots$ from I . Clearly, β is bi-additive: $\beta(p_1 + p_2, q) = \beta(p_1, q) + \beta(p_2, q)$ and $\beta(p, q_1 + q_2) = \beta(p, q_1) + \beta(p, q_2)$ for any $p, p_1, p_2, q, q_1, q_2 \in I$. Also, for any $p = a_1x + a_2y + a_3xy + \dots \in I$, $b_1x + b_2y + b_3xy + \dots \in I$, and $r = c_0 + c_1x + c_2 + \dots \in R$ we have $rp = c_0a_1x + c_0a_2y + \dots$, so $\beta(rp, q) = c_0a_1b_2 = c_0\beta(p, q) = r\beta(p, q)$, and similarly $\beta(p, rq) = r\beta(p, q)$. Hence, β is R -bilinear, and induces an R -module homomorphism $\varphi: I \otimes_R I \rightarrow \mathbb{Z}$ satisfying $\varphi(p, q) = a_1b_2$ for all polynomials $p = a_1x + a_2y + a_3xy + \dots$ and $q = b_1x + b_2y + b_3xy + \dots$ from I . Since $\varphi(w) = \varphi(x \otimes y) - \varphi(y \otimes x) = 1 - 0 \neq 0$, it must be that $w \neq 0$.

10pt **A2.** If a short exact sequence $0 \rightarrow N \xrightarrow{\varphi} M \xrightarrow{\psi} K \rightarrow 0$ of modules splits from the left (that is, there is a homomorphism $\tau: M \rightarrow N$ such that $\tau \circ \varphi = \text{Id}_N$) prove that $M = N' \oplus K'$ where $N' = \varphi(N)$ is isomorphic to N (under φ) and $K' = \ker \tau$ is isomorphic to K under $\psi|_{K'}$.

Solution.

$$0 \rightarrow N \xrightleftharpoons[\tau]{\varphi} M \xrightarrow{\psi} K \rightarrow 0$$

If $u \in K' \cap N' = \ker \tau \cap \varphi(N)$, with $u = \varphi(v)$, $v \in N$, then $\tau(u) = 0$ and $\tau(\varphi(v)) = v$, so $v = 0$, so $u = 0$. Hence, $K' \cap N' = 0$.

Next, for every $u \in M$, let $v = \tau(u) \in N$ and $u_1 = \varphi(v) \in N'$. Then $\tau(u - u_1) = v - \tau(\varphi(v)) = v - v = 0$, so $u_2 = u - u_1 \in \ker \tau = K'$, and $u = u_1 + u_2$. Hence, $M = N' + K' = N' \oplus K'$.

Since $N' = \ker \psi$, it also follows that $\psi|_{K'}$ is injective and surjective, that is, is an isomorphism between K' and K .

Another solution. Consider the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & N & \xrightleftharpoons[\tau]{\varphi} & M & \xrightarrow{\psi} & K \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & N & \rightarrow & N \oplus K & \rightarrow & K \rightarrow 0 \end{array}$$

By the universal property of the direct sum (or rather direct product), the homomorphisms $\tau: M \rightarrow N$ and $\psi: M \rightarrow K$ define a homomorphism $M \rightarrow N \oplus K$ that makes the diagram (excluding τ) commutative. By the short five lemma, it's an isomorphism, so $M = \varphi(N) \oplus K'$ for a copy K' of K .

If, using this isomorphism, we identify M and $N \oplus K$, then φ is the natural embedding $N \rightarrow N \oplus K$ and ψ is the natural projection $N \oplus K \rightarrow K$. τ is a homomorphism for which φ is a section, so τ is the natural projection $N \oplus K \rightarrow N$. Hence, $K = \ker \tau = K'$, and $\psi|_{K'}$ is an isomorphism $K' \rightarrow K$.

10pt **A3.** Let A be a unital R -algebra and M be an A -module. By reducing scalars, consider M as an R -module (with $au = (a1_A)u$). Prove that the R -module homomorphism $\varphi: M \rightarrow A \otimes_R M$ defined by $\varphi(u) = 1_A \otimes u$ is injective and that $1_A \otimes M = \varphi(M)$ is a direct summand in $A \otimes_R M$.

Solution. To prove both assertions it suffices to find an R -module homomorphism $\pi: A \otimes_R M \rightarrow M$ such that $\pi \circ \varphi = \text{Id}_M$: if such π exists, then φ is injective, and π is a left splitting homomorphism of the short exact sequence

$$0 \rightarrow M \xrightarrow{\varphi} A \otimes_R M \rightarrow (A \otimes_R M)/\varphi(M) \rightarrow 0.$$

But this is easy: simply put $\pi(\alpha \otimes u) = \alpha u$, $\alpha \in A$, $u \in M$. (Well, to prove that this π is well defined, we need to start with a bilinear mapping $A \times M \rightarrow M$, etc.)

A4. If $\varphi_1: M_1 \rightarrow N_1$ and $\varphi_2: M_2 \rightarrow N_2$ are two homomorphisms of R -modules, then the homomorphism $\varphi_1 \otimes \varphi_2: M_1 \otimes M_2 \rightarrow N_1 \otimes N_2$ is defined by $(\varphi_1 \otimes \varphi_2)(u_1 \otimes u_2) = \varphi_1(u_1) \otimes \varphi_2(u_2)$. However, the same notation $\varphi_1 \otimes \varphi_2$ applies to an element of the tensor product $\text{Hom}(M_1, N_1) \otimes \text{Hom}(M_2, N_2)$.

5pt (a) Prove that there is a unique homomorphism

$$\text{Hom}(M_1, N_1) \otimes \text{Hom}(M_2, N_2) \rightarrow \text{Hom}(M_1 \otimes M_2, N_1 \otimes N_2)$$

that maps $\varphi_1 \otimes \varphi_2$ in the second sense to $\varphi_1 \otimes \varphi_2$ in the first sense.

Solution. Let's denote $\varphi_1 \otimes \varphi_2$ "in the first sense" by $\varphi_1 \otimes_1 \varphi_2$ and $\varphi_1 \otimes \varphi_2$ "in the second sense" by $\varphi_1 \otimes_2 \varphi_2$. The mapping $\text{Hom}(M_1, N_1) \times \text{Hom}(M_2, N_2) \rightarrow \text{Hom}(M_1 \otimes M_2, N_1 \otimes N_2)$, $(\varphi_1, \varphi_2) \mapsto \varphi_1 \otimes_1 \varphi_2$ is bilinear. (Indeed, for $\varphi_1, \psi_1 \in \text{Hom}(M_1, N_1)$, $\varphi_2 \in \text{Hom}(M_2, N_2)$, and any $u_1 \in M_1$, $u_2 \in M_2$ we have

$$\begin{aligned} ((\varphi_1 + \psi_1) \otimes_1 \varphi_2)(u_1 \otimes u_2) &= (\varphi_1 + \psi_1)(u_1) \otimes \varphi_2(u_2) = \varphi_1(u_1) \otimes \varphi_2(u_2) + \psi_1(u_1) \otimes \varphi_2(u_2) \\ &= (\varphi_1 \otimes_1 \varphi_2)(u_1 \otimes u_2) + (\psi_1 \otimes_1 \varphi_2)(u_1 \otimes u_2), \end{aligned}$$

etc.) So, it induces a unique homomorphism $\text{Hom}(M_1, N_1) \otimes \text{Hom}(M_2, N_2) \rightarrow \text{Hom}(M_1 \otimes M_2, N_1 \otimes N_2)$ that maps $\varphi_1 \otimes_2 \varphi_2 \mapsto \varphi_1 \otimes_1 \varphi_2$.

5pt (b) Find an example where the homomorphism in (a) is not an isomorphism.

Solution. Take $R = \mathbb{Z}$, $M_1 = N_2 = \mathbb{Z}$, $M_2 = N_1 = \mathbb{Z}_2$. Then $\text{Hom}(M_1, N_1) \cong \mathbb{Z}_2$, $\text{Hom}(M_2, N_2) = 0$, so $\text{Hom}(M_1, N_1) \otimes \text{Hom}(M_2, N_2) = 0$, whereas $M_1 \otimes M_2 \cong N_1 \otimes N_2 \cong \mathbb{Z}_2$ and so $\text{Hom}(M_1 \otimes M_2, N_1 \otimes N_2) \cong \mathbb{Z}_2$.

Another solution. Take $R = \mathbb{Z}$, $M_1 = N_1 = M_2 = \mathbb{Z}_2$ and $N_2 = \mathbb{Q}/\mathbb{Z}$. Then $\text{Hom}(M_1, N_1) \cong \mathbb{Z}_2$, $\text{Hom}(M_2, N_2) \cong \mathbb{Z}_2$ (1 can be sent to either 0 or 1/2), so $\text{Hom}(M_1, N_1) \otimes \text{Hom}(M_2, N_2) \cong \mathbb{Z}_2$. However, $N_1 \otimes N_2 = 0$; indeed, \mathbb{Q}/\mathbb{Z} is "divisible": for any $a \in \mathbb{Q}/\mathbb{Z}$ there is b such that $a = 2b$, so $1 \otimes a = 2 \otimes b = 0$. So, $\text{Hom}(M_1 \otimes M_2, N_1 \otimes N_2) = 0$.

A5. Let A be a graded algebra.

10pt (a) Prove that a (two-sided) ideal I in A is graded iff for every $u \in I$, all homogeneous components of u are also contained in I .

Solution. Let I be graded, let $u \in I$. Since $I = \bigoplus_{i=0}^{\infty} I_i$, $u = \sum_{i=0}^{\text{fin}} u_i$ with $u_i \in I_i \subseteq A_i$ for every i . So, u_i are the homogeneous components of u , and are contained in I .

Conversely, assume that for every $u \in I$ all homogeneous components of u are contained in I . Then u is the sum of these components, and so, is contained in $\sum_{i=0}^{\infty} (I \cap A_i) = \sum_{i=0}^{\infty} I_i$. Hence, $I \subseteq \sum_{i=0}^{\infty} I_i$; the opposite inclusion $\sum_{i=0}^{\infty} I_i \subseteq I$ is evident. So, $I = \sum_{i=0}^{\infty} I_i$; the sum is direct since for any i , $(I_i \cap \sum_{j \neq i} I_j) \subseteq (A_i \cap \sum_{j \neq i} A_j) = 0$.

5pt (b) Prove that an ideal I in A is graded iff it is homogeneous.

Solution. If an ideal is graded, then it is generated by homogeneous elements even as a module. Assume conversely that an ideal I is generated, as an ideal, by a set S of homogeneous elements. Then as a fin module I is generated by elements of the form $\alpha s \beta$ for $\alpha, \beta \in A$, $s \in S$. For any such α, β and s , let $\alpha = \sum_{i=0}^{\text{fin}} \alpha_i$ and $\beta = \sum_{i=0}^{\text{fin}} \beta_i$ where α_i and β_i are homogeneous; then $\alpha s \beta = \sum_{i,j=0}^{\text{fin}} \alpha_i s \beta_j$, and for any i and j , $\alpha_i s \beta_j$ is a homogeneous element of I . Hence, every element of I is a sum of homogeneous elements of I , so I is graded by (a).

5pt (c) If I is a graded ideal in A , prove that A/I has a structure of a graded algebra.

Solution. Let I be graded, $I = \bigoplus_{i=0}^{\infty} I_i$ with $I_i \subseteq A_i$ for all i . Define an R -module homomorphism $\varphi: A \rightarrow \bigoplus_{i=0}^{\infty} A_i/I_i$ by $\varphi(u_0, u_1, u_2, \dots) = (u_0 \text{ mod } I_0, u_1 \text{ mod } I_1, u_2 \text{ mod } I_2, \dots)$. φ is surjective and $\ker(\varphi) = \bigoplus_{i=0}^{\infty} I_i = I$, so induces an isomorphism $A/I \rightarrow \bigoplus_{i=0}^{\infty} A_i/I_i$. And $\bigoplus_{i=0}^{\infty} A_i/I_i$ is a "graduation" of A/I : for any $s \in A_i/I_i$ and $t \in A_j/I_j$, $p = \varphi(u)$ and $q = \varphi(v)$ for $u \in A_i$ and $v \in A_j$, so $pq = \varphi(u)\varphi(v) = \varphi(uv)$ where $uv \in A_{i+j}$, and so $pq \in A_{i+j}/I_{i+j}$.

5pt **A6.** Let M be an R -module, $u_1, u_2 \in M$, $a_1, a_2, b_1, b_2 \in R$, $v_1 = a_1 u_1 + a_2 u_2$ and $v_2 = b_1 u_1 + b_2 u_2$. Find $c \in R$ such that in $\Lambda^2(M)$, $v_1 \wedge v_2 = c u_1 \wedge u_2$.

Solution.

$$\begin{aligned} v_1 \wedge v_2 &= (a_1 u_1 + a_2 u_2) \wedge (b_1 u_1 + b_2 u_2) = a_1 b_1 u_1 \wedge u_1 + a_1 b_2 u_1 \wedge u_2 + b_1 a_2 u_2 \wedge u_1 + b_1 b_2 u_2 \wedge u_2 \\ &= a_1 b_2 u_1 \wedge u_2 - b_1 a_2 u_1 \wedge u_2 = (a_1 b_2 - a_2 b_1) u_1 \wedge u_2. \end{aligned}$$