In all problems, all modules are assumed to be over a commutative unital ring R.

5pt **10.5.22.** *Prove that the tensor product of two flat modules is flat.*

Solution. Let K_1 and K_2 be flat modules and let $\varphi\colon M \longrightarrow N$ be a monomorphism. Then $\varphi \otimes \mathrm{Id}_{K_1}\colon M \otimes K_1 \longrightarrow N \otimes K_1$ is a monomorphism, and then $(\varphi \otimes \mathrm{Id}_{K_1}) \otimes \mathrm{Id}_{K_2}\colon M \otimes K_1 \otimes K_2 \longrightarrow N \otimes K_1 \otimes K_2$ is a monomorphism. So, $K_1 \otimes K_2$ is flat.

10.5.23. *Let S be a commutative unital R-algebra.*

10pt (a) *If K is an R-module and M is an S-module, prove that $M \otimes_R K \cong M \otimes_S (S \otimes_R K)$ as S-modules.*

Solution. It would be easy to write $M \otimes_S (S \otimes_R K) \cong (M \otimes_S S) \otimes_R K \cong M \otimes_R K$, but it wouldn't be fair, since we've never proven the "associativity" for tensor products over two distinct rings.

The isomorphism is defined by $u \otimes (\alpha \otimes v) = (\alpha u) \otimes v$, $u \in M$, $\alpha \in S$, $v \in K$. Indeed, for any $u \in M$, the mapping $S \times K \longrightarrow M \otimes_R K$ defined by $(\alpha, v) \mapsto (\alpha u) \otimes v$ is R-bilinear, so defines an R-module homomorphism $\varphi_u\colon S \otimes_R K \longrightarrow M \otimes_R K$ with $\varphi_u(\alpha \otimes v) \mapsto (\alpha u) \otimes v$, $u \in M$, $\alpha \in S$, $v \in K$. Next, the mapping $M \times (S \otimes_R K) \longrightarrow M \otimes_R K$ defined by $(u, \omega) \mapsto \varphi_u(\omega)$, $u \in M$, $\omega \in S \otimes_R K$, is S-bilinear. (Indeed, for any $u_1, u_2 \in M$, $\alpha \in S$, and $v \in K$ we have

$$(u_1 + u_2, \alpha \otimes v) \mapsto \varphi_{u_1+u_2}(\alpha \otimes v) = (\alpha(u_1 + u_2)) \otimes v = (\alpha u_1) \otimes v + (\alpha u_2) \otimes v,$$

so $\varphi_{u_1+u_2}(\omega) = \varphi_{u_1}(\omega) + \varphi_{u_2}(\omega)$ for all $\omega \in S \otimes_R K$, and for any $u \in M$, $\alpha, \beta \in S$, and $v \in K$,

$$(\beta u, \alpha \otimes v) \mapsto \varphi_{\beta u}(\alpha \otimes v) = (\alpha\beta u) \otimes v = \beta((\alpha u) \otimes v),$$

so $\varphi_{\beta u}(\omega) = \beta\varphi_u(\omega)$ for all $\omega \in S \otimes_R K$. Also, for any $u \in M$ and $\omega_1, \omega_2 \in S \otimes_R K$,

$$(u, \omega_1 + \omega_2) \mapsto \varphi_u(\omega_1 + \omega_2) = \varphi_u(\omega_1) + \varphi_u(\omega_2)$$

and for any $u \in M$, $\alpha \in S$, and $v \in K$,

$$(u, \beta(\alpha \otimes v)) = (u, (\beta\alpha) \otimes v) \mapsto \varphi_u((\beta\alpha) \otimes v) = (\beta\alpha u) \otimes v = \beta((\alpha u) \otimes v) = \beta\varphi_u(\alpha \otimes v),$$

so $(u, \beta\omega) \mapsto \beta\varphi_u(\omega)$ for all $\omega \in S \otimes_R K$.) Hence, this mapping induces an S-module homomorphism $\varphi\colon M \otimes_S (S \otimes_R K) \longrightarrow M \otimes_R K$ with $\varphi(u \otimes (\alpha \otimes v)) = (\alpha u) \otimes v$ for all $u \in M$, $\alpha \in S$, $v \in K$. The inverse homomorphism $\psi\colon M \otimes_R K \longrightarrow M \otimes_S (S \otimes_R K)$ is defined by $\psi(u \otimes v) = u \otimes (1 \otimes v)$, $u \in M$, $v \in N$.

10pt (b) *If K is a flat R-module prove that $S \otimes_R K$ is a flat S-module.*

Solution. Let $\varphi\colon M \longrightarrow N$ be a monomorphism or S-modules. Then M and N are also R-modules; since K is flat, the homomorphism $\varphi \otimes_R \mathrm{Id}_K\colon M \otimes_R K \longrightarrow N \otimes_R K$ is also injective. The isomorphism in (a) is "functorial", in the sense that the diagram

$$\begin{array}{ccc} M \otimes_S (S \otimes_R K) & \overset{\varphi \otimes_S \mathrm{Id}_{S \otimes_R K}}{\longrightarrow} & N \otimes_S (S \otimes_R K) \\ \updownarrow & & \updownarrow \\ M \otimes_R K & \overset{\varphi \otimes_R \mathrm{Id}_K}{\longrightarrow} & N \otimes_R K \end{array}$$

is commutative. (Indeed, along two ways from the upper left corner to the lower right corner, for any $u \in M$, $\alpha \in S$, $v \in K$,

$$u \otimes (\alpha \otimes v) \mapsto \varphi(u) \otimes (\alpha \otimes v) \mapsto (\alpha\varphi(u)) \otimes v = \varphi(\alpha u) \otimes v$$

and

$$u \otimes (\alpha \otimes v) \mapsto (\alpha u) \otimes v \mapsto \varphi(\alpha u) \otimes v.)$$

Hence, the homomorphism $\varphi \otimes_S \mathrm{Id}_{S \otimes_R K}\colon M \otimes_S (S \otimes_R K) \longrightarrow N \otimes_S (S \otimes_R K)$ is also injective, so, $S \otimes_R K$ is a flat S-module.

5pt **A1.** *Let $R = \mathbb{Z}[x, y]$ and let I be the ideal (x, y) in R. Prove that as an R-module, I is torsion-free but not flat.*

Solution. I is a submodule of an intergal domain, so it is torsion free; but, as we know, $\mathrm{Tor}(I \otimes I) \neq 0$. ($\omega = x \otimes y - y \otimes x$ is a nonzero torsion element of $I \otimes I$). So, the homomorphism $I \otimes I \longrightarrow I \otimes \mathbb{R} \cong I$ is not injective (all torsion elements go to 0).

5pt **10.5.9(b).** *Prove that the tensor product of two projective modules is projective.*

Solution. Let P_1 and P_2 be projective, let (by the criterion of projectivity) P_1' and P_2' be such that $P_1 \oplus P_1'$ and $P_2 \oplus P_2'$ are free. Since the tensor product of free modules is free,

$$(P_1 \oplus P_1') \otimes (P_2 \oplus P_2') \cong (P_1 \otimes P_2) \oplus (P_1' \otimes P_2) \oplus (P_1 \otimes P_2') \oplus (P_1' \otimes P_2')$$

is also free, so $P_1 \otimes P_2$ is a direct summand of a free module, and so, is projective.

5pt **A2.** *Prove that a module P is projective iff it "splits from the right any short exact sequence", that is, iff every epimorphism $\pi\colon M \longrightarrow P$ has a section (a homomorphism $\sigma\colon P \longrightarrow M$ such that $\pi\circ\sigma = \mathrm{Id}_P$).*

Solution. Let P be projective and let $\pi\colon M \longrightarrow P$ be an epimorphism. Then the identity homomorphism Id_P can be lifted to a homomorphism $\sigma\colon P \longrightarrow M$ such that $\mathrm{Id}_P = \pi\circ\sigma$, that is, σ is a section of π.

Conversely, assume that any epimorphism $M \longrightarrow P$ has a section. Find a free module M with an epimorphism $\pi\colon M \longrightarrow P$; then the exact seqeunce $0 \longrightarrow \ker(\pi) \longrightarrow M \xrightarrow{\pi} P \longrightarrow 0$ splits, so $M = P \oplus \ker(\pi)$, so P is a direct summand of a free module, so P is projective.

5pt **A3.** *If F is a field, prove that every F-module is flat, projective, and injective.*

Solution. Every F-vector space is a free module, thus it is flat and projective.

Let L be an F-vector space. Let W be a subspace of an F-vector space V, and let φ be a homomorphism (linear mapping) $W \longrightarrow L$. Since W is projective, it is a direct summand of V, $V = W \oplus W'$ for some subspace W', and we can extend φ to a homomorphism $V \longrightarrow L$ by putting $\varphi_{|W'} = 0$.

5pt **A4.** *Prove that an integral domain R is an injective R-module iff R is a field.*

Solution. If R is a field, R is injective since it is its own field of fractions; or by A3.

If R is injective, then R is divisible, and so, for every nonzero $a \in R$ there exists $b \in R$ such that $ba = 1$, that is, a is a unit.

5pt **10.5.4.** *Given modules Q_1 and Q_2, prove that $Q = Q_1 \oplus Q_2$ is injective iff both Q_1 and Q_2 are injective.*

Solution. Assume that Q_1 and Q_2 are injective, let N be a submodule of a module M, and let φ be a homomorphism $N \longrightarrow Q$. Let φ_1 and φ_2 be the components of φ: $\varphi = (\varphi_1, \varphi_2)$ let $\psi_1\colon M \longrightarrow Q_1$ and $\psi_2\colon M \longrightarrow Q_2$ be extensions of φ_1 and φ_2, and let $\psi = (\psi_1, \psi_2)\colon M \longrightarrow Q$. Then ψ is an extension of φ, which proves that Q is injective.

Conversely, assume that Q is injective, let N be a submodule of a module M, and let φ be a homomorphism $N \longrightarrow Q_1$. Extend the homomorphism $(\varphi, 0)\colon N \longrightarrow Q$ to a homomorphism $\psi\colon M \longrightarrow Q$, let $\psi = (\psi_1, \psi_2)$. Then ψ_1 is a homomorphism $M \longrightarrow Q_1$ which coincides with φ on N, which proves that Q_1 is injective.

Another solution. For any module M, $\mathrm{Hom}(M, Q) \cong \mathrm{Hom}(M, Q_1) \oplus \mathrm{Hom}(M, Q_2)$, where $\varphi\colon M \longrightarrow Q$ corresponds to the pair (φ_1, φ_2) of its "components" $\varphi_1\colon M \longrightarrow Q_1$ and $\varphi_2\colon M \longrightarrow Q_2$. This isomorphism is functorial: given a homomorphism $\tau\colon M \longrightarrow N$ to another module, the induced homomorphism $\mathrm{Hom}(N, Q) \longrightarrow \mathrm{Hom}(M, Q)$, $\varphi \mapsto \varphi\circ\tau$, commutes with the decomposition above and the similar decomposition of $\mathrm{Hom}(N, Q)$: if $\varphi = (\varphi_1, \varphi_2)$, then $\varphi\circ\tau = (\varphi_1\circ\tau, \varphi_2\circ\tau)$.

Now, given an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, the seqeunce $0 \longrightarrow \mathrm{Hom}(C, Q) \longrightarrow \mathrm{Hom}(B, Q) \longrightarrow \mathrm{Hom}(A, Q) \longrightarrow 0$ is exact iff the both its "component" sequences $0 \longrightarrow \mathrm{Hom}(C, Q_1) \longrightarrow \mathrm{Hom}(B, Q_1) \longrightarrow \mathrm{Hom}(A, Q_1) \longrightarrow 0$ and $0 \longrightarrow \mathrm{Hom}(C, Q_2) \longrightarrow \mathrm{Hom}(B, Q_2) \longrightarrow \mathrm{Hom}(A, Q_2) \longrightarrow 0$ are exact; thus Q is injective iff both Q_1 and Q_2 are.

10.5.12. *Let M_α, $\alpha \in \Lambda$, be a family of R-modules.*

5pt (a) *Prove that for any module N, $\mathrm{Hom}\big(N, \prod_{\alpha\in\Lambda} M_\alpha\big) \cong \prod_{\alpha\in\Lambda} \mathrm{Hom}(N, M_\alpha)$ as R-modules.*

Solution. A homomorphism $\varphi\colon N \longrightarrow \prod_{\alpha\in\Lambda} M_\alpha$ induces homomorphisms $\varphi_\alpha\colon N \longrightarrow M_\alpha$, $\alpha \in \Lambda$, by $\varphi_\alpha = \pi_\alpha\circ\varphi$, where π_α are the projections $\prod_{\alpha\in\Lambda} M_\alpha \longrightarrow M_\alpha$; this gives a mapping $\Phi\colon \mathrm{Hom}\big(N, \prod_{\alpha\in\Lambda} M_\alpha\big) \longrightarrow \prod_{\alpha\in\Lambda} \mathrm{Hom}(N, M_\alpha)$. Conversely, any collection of homomorphisms $\varphi_\alpha\colon N \longrightarrow M_\alpha$, $\alpha \in \Lambda$, defines a homomorphism $\varphi\colon N \longrightarrow \prod_{\alpha\in\Lambda} M_\alpha$ by $\varphi(u) = \big(\varphi_\alpha(u),\ \alpha \in \Lambda\big)$; this mapping $\prod_{\alpha\in\Lambda} \mathrm{Hom}(N, M_\alpha) \longrightarrow \mathrm{Hom}\big(N, \prod_{\alpha\in\Lambda} M_\alpha\big)$ is the inverse of Φ. Hence, Φ is a bijection; it is only to prove that Φ is a homomorphism. And indeed, for any $\varphi, \psi\colon N \longrightarrow \prod_{\alpha\in\Lambda} M_\alpha$ for any $\alpha \in \Lambda$ we have $(\varphi + \psi)_\alpha = \pi_\alpha\circ(\varphi + \psi) = \pi_\alpha\circ\varphi + \pi_\alpha\circ\psi = \varphi_\alpha + \psi_\alpha$ and for any $a \in R$ for any $\alpha \in \Lambda$ we have $(a\varphi)_\alpha = \pi_\alpha\circ(a\varphi) = a(\pi_\alpha\circ\varphi) = a\varphi_\alpha$.

5pt (b) *Prove that for any module N, $\mathrm{Hom}\big(\bigoplus_{\alpha\in\Lambda} M_\alpha, N\big) \cong \prod_{\alpha\in\Lambda} \mathrm{Hom}(M_\alpha, N)$ as R-modules. In particular, $\big(\bigoplus_{\alpha\in\Lambda} M_\alpha\big)^* \cong \prod_{\alpha\in\Lambda} M_\alpha^*$.*

Solution. A homomorphism $\varphi\colon \bigoplus_{\alpha\in\Lambda} M_\alpha \longrightarrow N$ induces homomorphisms $\varphi_\alpha\colon M_\alpha \longrightarrow N$, $\alpha \in \Lambda$, by $\varphi_\alpha = \varphi|_{M_\alpha}$; this gives a mapping $\Phi\colon \mathrm{Hom}\big(\bigoplus_{\alpha\in\Lambda} M_\alpha, N\big) \longrightarrow \prod_{\alpha\in\Lambda} \mathrm{Hom}(M_\alpha, N)$. Conversely, any collection of homomorphisms $\varphi_\alpha\colon M_\alpha \longrightarrow N$, $\alpha \in \Lambda$, defines a homomorphism $\varphi\colon \bigoplus_{\alpha\in\Lambda} M_\alpha \longrightarrow N$ by $\varphi\big(\sum^{\text{fin}}_{\alpha\in\Lambda} u_\alpha\big) = \sum^{\text{fin}}_{\alpha\in\Lambda} \varphi_\alpha(u_\alpha)$; this mapping $\prod_{\alpha\in\Lambda} \mathrm{Hom}(M_\alpha, N) \longrightarrow \mathrm{Hom}\big(\bigoplus_{\alpha\in\Lambda} M_\alpha, N\big)$ is the inverse of Φ. Hence, Φ is a bijection; it is only to prove that Φ is a homomorphism. And indeed, for any $\varphi, \psi\colon \bigoplus_{\alpha\in\Lambda} M_\alpha \longrightarrow N$ for any $\alpha \in \Lambda$ we have $(\varphi + \psi)_\alpha = (\varphi + \psi)|_{M_\alpha} = \varphi|_{M_\alpha} + \psi|_{M_\alpha} = \varphi_\alpha + \psi_\alpha$ and for any $a \in R$ for any $\alpha \in \Lambda$, $(a\varphi)_\alpha = (a\varphi)|_{M_\alpha} = a(\varphi|_{M_\alpha}) = a\varphi_\alpha$.