In all problems, all modules are assumed to be over a commutative unital ring R.

5pt **A1.** (a) *For a subspace W of a finite dimensional vector space V, prove that $W = \mathrm{Ann}(\mathrm{Ann}(W))$. Deduce that for two subspaces W_1 and W_2 of V one has $\mathrm{Ann}(W_1) = \mathrm{Ann}(W_2)$ iff $W_1 = W_2$.*

Solution. We have $W \subseteq \mathrm{Ann}(\mathrm{Ann}(W))$ for any module W. In the case of finite dimensional vector spaces, we also have that $V^{**} = V$, so $\dim(\mathrm{Ann}(\mathrm{Ann}(W)) = \dim V - \dim(\mathrm{Ann}\, W) = \dim W$, which implies that, in fact, $W = \mathrm{Ann}(\mathrm{Ann}(W))$.

If $\mathrm{Ann}(W_1) = \mathrm{Ann}(W_2)$, then $W_1 = \mathrm{Ann}(\mathrm{Ann}(W_1)) = \mathrm{Ann}(\mathrm{Ann}(W_2)) = W_2$.

Another solution. This proof doesn't use the assumption that $\dim V < \infty$. Let $u \in V \setminus W$; put $f|_W = 0$ and $f(u) = 1$, and extend f to an element V^*. Then $f \in \mathrm{Ann}(W)$, $u(f) = f(u) \neq 0$, so $u \notin \mathrm{Ann}(\mathrm{Ann}(f))$.

5pt (b) *If W_1 and W_2 are two subspaces of a finite dimensional vector space V, prove that $\mathrm{Ann}(W_1 \cap W_2) = \mathrm{Ann}(W_1) + \mathrm{Ann}(W_2)$.*

Solution. Since (a) holds for infinite dimensional spaces also, the assumption that $\dim V < \infty$ is redundant. For $U_1 = \mathrm{Ann}(W_1)$ and $U_2 = \mathrm{Ann}(W_2)$ we have, by (a), that $W_1 = \mathrm{Ann}(U_1)$ and $W_2 = \mathrm{Ann}(U_2)$. Since $W_1 \cap W_2 = \mathrm{Ann}(U_1 + U_2)$, by (a) again, $\mathrm{Ann}(W_1 \cap W_2) = U_1 + U_2 = \mathrm{Ann}(W_1) + \mathrm{Ann}(W_2)$.

Another solution. This proof also doesn't use the assumption that $\dim V < \infty$. We have $\mathrm{Ann}(W_1) + \mathrm{Ann}(W_2) \subseteq \mathrm{Ann}(W_1 \cap W_2)$; we need to prove the opposite inclusion. Let $W = W_1 \cap W_2$; find subspaces L_1, L_2, W' such that $W_1 = W \oplus L_1$, $W_2 = W \oplus L_2$, and $V = (W_1 + W_2) \oplus W'$; then $V = W \oplus L_1 \oplus L_2 \oplus W'$.

Now let $f \in \mathrm{Ann}(W_1 \cap W_2)$; define $f_1, f_2 \in V^*$ by $f_1|_W = 0$, $f_1|_{L_1} = f$, $f_1|_{L_2} = 0$, $f_1|_{W'} = f$ and $f_2|_W = 0$, $f_2|_{L_2} = f$, $f_2|_{L_2} = 0$, $f_2|_{W'} = 0$. Then $f_1 \in \mathrm{Ann}(W_2)$, $f_2 \in \mathrm{Ann}(W_1)$, and $f_1 + f_2 = f$, so $f \in \mathrm{Ann}(W_1) + \mathrm{Ann}(W_2)$.

5pt **A2.** *If M and N are free modules of fintie rank, prove that the natural homomorphism $M^* \otimes N^* \longrightarrow (M \otimes N)^*$, $f \otimes g \mapsto h$ defined by $h(u \otimes v) = f(u)g(v)$, is an isomorphism.*

Solution. Let $\{u_1, \ldots, u_n\}$ be a basis in M and $\{f_1, \ldots, f_n\}$ be the dual basis in M^*; let $\{v_1, \ldots, v_m\}$ be a basis in N, and $\{g_1, \ldots, g_m\}$ be the dual basis in N^*. Then $\{u_i \otimes v_j,\ i = 1, \ldots, n,\ j = 1, \ldots, m\}$ is a basis in $M \otimes N$ and $\{f_i \otimes g_j,\ i = 1, \ldots, n,\ j = 1, \ldots, m\}$ is a basis in $M^* \otimes N^*$. The image $h_{i,j}$ of $f_i \otimes g_j$ in $(M \otimes N)^*$ acts on the tensor $u_k \otimes v_l$ in the following way: $h_{i,j}(u_k \otimes v_l) = f_i(u_k)g_j(v_l) = 1$ if $i = k$, $j = l$ and $= 0$ otherwise. Hence, $\{h_{i,j},\ i = 1, \ldots, n,\ j = 1, \ldots, m\}$ is the dual basis in $(M \otimes N)^*$ of the basis $\{u_i \otimes v_j,\ i = 1, \ldots, n,\ j = 1, \ldots, m\}$ in $M \otimes N$. A homomorphism that bijectively maps a basis onto a basis is an isomorphism, so $M^* \otimes N^* \longrightarrow (M \otimes N)^*$.

10pt **A3.** *Let $\varphi\colon M \longrightarrow N$ be a homomorphism of two free modules of finite rank and let ω be the tensor corresponding to φ in $N \otimes M^*$. Prove that $\mathrm{rank}\,\varphi \leq \mathrm{rank}\,\omega$, and if R is a field, then $\mathrm{rank}\,\varphi = \mathrm{rank}\,\omega$.*

Sorry, it is my mistake: We have to assume that R is an integral domain, otherwise the rank of a non-free module is not well defined.

Solution. Let $\omega \in N \otimes M^*$ be the tensor that corresponds to φ. Let tensor-rank of ω be l. Write $\omega = \sum_{i=1}^{l} v_i \otimes f_i$, let L be the submodule of N generated by $v_1, \ldots, v_l$, then $\mathrm{rank}\, L \leq l$. For any $u \in M$ we have $\varphi(u) = \sum_{i=1}^{k} f_i(u)v_i \in L$, so $\varphi(M) \subseteq L$, so $\mathrm{rank}\,\varphi = \mathrm{rank}(\varphi(M)) \leq l = \mathrm{rank}\,\omega$.

Now let R be a field (so that M and N are vector spaces), let $k = \mathrm{rank}\,\varphi$, let $K = \varphi(M)$, and let $\{v_1, \ldots, v_k\}$ be a basis of K. Since M^* is a flat module, $K \otimes M^*$ is a submodule of $N \otimes M^*$, and $\omega \in K \otimes M^*$. So, $\omega = \sum_{i=1}^{k} v_i \otimes f_i$ for some $f_1, \ldots, f_k \in M^*$, so $\mathrm{rank}\,\omega \leq k = \mathrm{rank}\,\varphi$.

5pt **A4.** *Let M be a free module of finite rank, and let P be the transition matrix from a basis B to a basis C in M. Prove that the transition matrix from the basis B^* to the basis C^* in M^* is $(P^T)^{-1}$. (B^* and C^* are the dual bases of B and C respectively.)*

Solution. If φ is the "change of coordinates homomorphism" $R^n \longrightarrow R^n$ from basis B to basis C in M, then φ^* is the dual "change of coordinates homomorphism" $R^n \longrightarrow R^n$ from C^* to B^* in M^*. If P is the matrix of φ, then the matrix of φ^* is P^T, and the transition matrix from B^* to C^* is $(P^T)^{-1}$.

5pt **11.3.4.** *If M is a free module of an infnite rank with (an infnite) basis B, prove that B^* does not generate M^*.*

Solution. B^* only generates $\bigoplus_{u \in B} M^*$ whereas $M^* = \left(\bigoplus_{u \in B} M\right)^* = \prod_{u \in B} M^*$.

Or: just notice that the linear form $f \in M^*$ defined by $f(u) = 1$ for all $u \in B$ is not a finite linear combination of the forms f_u, $u \in B$.

5pt **A5.** (a) *If R is an integral domain, prove that every finitely generated R-module has finite rank.*

Solution. If M is a finitely generated R-module, we have an epimorphism $\varphi\colon R^n \longrightarrow M$, and then $\operatorname{rank} M \le n$. (Indeed, $n = \operatorname{rank} R^n = \operatorname{rank} M + \operatorname{rank}(\ker(\varphi))$.)

5pt (b) *Give an example of a non-finitely generated module of finite rank.*

Solution. It's $\mathbb{Q}$ as a $\mathbb{Z}$-module: $\operatorname{rank}_{\mathbb{Z}} \mathbb{Q} = 1$, but $\mathbb{Q}$ is not finitely generated. (Indeed, any $\frac{n_1}{m_1}, \dots, \frac{n_k}{m_k}$ are contained in the proper submodule $\frac{1}{m}\mathbb{Z}$ where $m = m_1 \cdots m_k$ of $\mathbb{Q}$, and so, cannot generate $\mathbb{Q}$.)

A6. *Let R be an ID, F be the field of fractions of R, M be an R-module, V be the F-vector space $F \otimes_R M$, $M^* = \operatorname{Hom}_R(M, R)$ and $V^* = \operatorname{Hom}_F(V, F)$.*

10pt (a) *For $f \in M^*$, define $\tilde{f} \in V^*$ by $\tilde{f}(\alpha \otimes u) = \alpha f(u)$; prove that $\tilde{f}$ is well defined. Prove that the mapping $f \mapsto \tilde{f}$ is a monomorphism (injective homomorphism) $M^* \longrightarrow V^*$. Deduce that the F-space homomorphism $F \otimes_R M^* \longrightarrow V^*$ defined by $1 \otimes f \mapsto \tilde{f}$ is also injective, and in the case* $\operatorname{rank} M < \infty$, *that* $\operatorname{rank} M^* \le \operatorname{rank} M$.

Solution. First, $\tilde{f}$ is well defined since the mapping $F \times M \longrightarrow F$, $(\alpha, u) \mapsto f(u)$, is bilinear. (Actually, $\tilde{f}$ is just the composition of the homomorphism $\operatorname{Id}_F \otimes f\colon F \otimes M \longrightarrow F \otimes R$ and the isomorphism $F \otimes R \longrightarrow F$, $\alpha \otimes a \mapsto \alpha a$.) It is also clear that the mapping $f \mapsto \tilde{f}$ is a homomorphism: $f_1 + f_2 \mapsto \tilde{f}_1 + \tilde{f}_2$ and $af \mapsto a\tilde{f}$. For any $u \in M$ we have $\tilde{f}(1 \otimes u) = f(u)$, so if $\tilde{f} = 0$ then $f = 0$; hence, the homomorphism $f \mapsto \tilde{f}$ has trivial kernel and so, is injective. Since F is a flat R-module, the induced homomorphism $F \otimes M^* \longrightarrow F \otimes V^* \cong V^*$ is also injective. If $\dim_F V = \operatorname{rank} M < \infty$ we have $\dim_F V^* = \dim_F V$, so $\operatorname{rank} M^* = \dim_F F \otimes M^* \le \dim_F V = \operatorname{rank} M$.

10pt (b) *If M is finitely generated, prove that the image of M^* in V^* spans (generates) V^*; moreover, for any $h \in V^*$ there exist $f \in M^*$ and $d \in R$ such that $dh = \tilde{f}$. Deduce that $F \otimes M^* \cong V^*$ and that* $\operatorname{rank} M^* = \operatorname{rank} M$.

Solution. Let M be generated by elements $u_1, \dots, u_n$. Let $h \in V^*$, and let d be a common denominator of the elements $h(1 \otimes u_1), \dots, h(1 \otimes u_n)$ of F, so that $dh(1 \otimes u_i) \in R$ for all i. Then $dh(1 \otimes u) \in R$ for all $u \in M$; put $f(u) = dh(1 \otimes u)$, then $f \in M^*$. Now $\tilde{f} = dh$, since for any $\alpha \in F$ and $u \in M$ we have $\tilde{f}(\alpha \otimes u) = \alpha f(u) = \alpha dh(1 \otimes u) = dh(\alpha \otimes u)$.

It follows that the image of M^* spans V^*, so the homomorphism $F \otimes M^* \longrightarrow V^*$ is surjective. Being injective by (a), it is an isomorphism. So, $\operatorname{rank} M^* = \dim_F(F \otimes M^*) = \dim_F V^* = \dim_F V = \operatorname{rank} M$.

5pt (c) *Give an example where for an (infinitely generated) R-module M of finite rank, the image of M^* doesn't span V^* (and so,* $\operatorname{rank} M^* < \operatorname{rank} M$*).*

Solution. Take $R = \mathbb{Z}$ and $M = \mathbb{Q}$, then $V \cong \mathbb{Q}$ and $V^* \cong \mathbb{Q}$, whereas $M^* = 0$. We have $\operatorname{rank} M = 1$ and $\operatorname{rank} M^* = 0$.