Solutions to Homework 7 **Math 5591H**

5pt **A1.** *If M is a finitely generated module over a PID, prove that M is flat iff M is projective iff M is torsion-free iff M is free.*

Solution. If M is free, then it is projective, if M is projective, then M is flat, and if M is flat, then M is torsion-free. However, for finitely generated modules of finite rank over a PID, being free and being torsion-free is the same.

A2. *Give an example of a module M over an integral domain R such that M is torsion-free but not free and*

5pt (a) *M is finitely generated but R is not a PID.*

Solution. $R = F[x, y]$, where F is a field, and $M = (x, y)$. Since $M/(x)$ is a torsion module, M has rank 1. However, M is not generated by a single element: for no $p(x, y) \in M$ both x and y are multiplies of p.

5pt (b) *R is a PID but M is not finitely generated.*

Solution. $R = \mathbb{Z}$, $M = \mathbb{Q}$. M has rank 1, but not generted by a single element (and is not, actually, finitely generated).

A3. *Let M be a free $\mathbb{Z}$-module of rank 4, $M_1 \cong M/N_1$ and $M_2 \cong M/N_2$, where N_1 is a subgroup (submodule) of M generated, in some basis $\{u_1, \dots, u_4\}$ of M, by $\{u_1, 2u_2, 6u_3\}$, and N_2 is a subgroup of M generated, in some basis $\{v_1, \dots, v_4\}$ of M, by $\{v_1, 3v_2, 6v_3\}$.*

5pt (a) *Are the modules N_1 and N_2 isomorphic?*

Solution. The vectors $u_1, 2u_2, 6u_3$ are linearly independent, so N_1 is a free $\mathbb{Z}$-module of rank 3. Similarly, N_2 is a free $\mathbb{Z}$-module of rank 3. Hence, $N_1 \cong N_2$.

5pt (b) *What are the ranks of the modules M_1 and M_2?*

Solution. $\operatorname{rank} M_1 = 4 - \operatorname{rank} N_1 = 1$ and $\operatorname{rank} M_2 = 4 - \operatorname{rank} N_2 = 1$.

5pt (c) *Are the modules M_1 and M_2 isomorphic?*

Solution. No: $M_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_6$ and has invariant factors 2 and 6; $M_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_6$ and has invariant factors 3 and 6.

A4. *Let N be the sublattice of $\mathbb{Z}^3$ generated by the vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\begin{pmatrix}-1\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\3\end{pmatrix}$, and let $M = \mathbb{Z}^3/N$.*

5pt (a) *Find the invariant factors of M.*

Solution. The relations matrix of M (a matrix, whose columns generate N) is $\begin{pmatrix}1&-1&1\\1&-1&-1\\1&1&3\end{pmatrix}$. Using row-column operations over $\mathbb{Z}$, we transform it this way:

$$\begin{pmatrix}1&-1&1\\1&-1&-1\\1&1&3\end{pmatrix} \mapsto \begin{pmatrix}1&-1&1\\0&0&-2\\0&2&2\end{pmatrix} \mapsto \begin{pmatrix}1&0&0\\0&0&-2\\0&2&2\end{pmatrix} \mapsto \begin{pmatrix}1&0&0\\0&2&2\\0&0&-2\end{pmatrix} \mapsto \begin{pmatrix}1&0&0\\0&2&0\\0&0&2\end{pmatrix}$$

and obtain that the invariant factors of M are $2, 2$.

5pt (b) *Determine the cardinality of M.*

Solution. From (a), it follows that $M \cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(2) = \mathbb{Z}_2^2$, and the cardinality of M is 4.

5pt **A5.** *Let G be an (additively written) abelian group defined by its generators u_1, u_2, u_3, u_4 and relations $2u_1 + 4u_2 + 10u_3 + 2u_4 = 0$ and $4u_1 = 2u_2 + 6u_4$. Represent G as a product of cyclic groups.*

Solution. The relations matrix of G is $\begin{pmatrix}2&4\\4&-2\\10&0\\2&-6\end{pmatrix}$. Using row-column operations over $\mathbb{Z}$, we transform it this way:

$$\begin{pmatrix}2&4\\4&-2\\10&0\\2&-6\end{pmatrix} \mapsto \begin{pmatrix}2&4\\0&-10\\0&-20\\0&-10\end{pmatrix} \mapsto \begin{pmatrix}2&0\\0&-10\\0&-20\\0&-10\end{pmatrix} \mapsto \begin{pmatrix}2&0\\0&10\\0&0\\0&0\end{pmatrix}.$$

Hence, the rank of G is 2, the invariant factors of G are 2 and 10, and $G \cong \mathbb{Z}^2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$.

5pt **12.2.4.** *Prove that 3×3 matrices over a field are similar iff they have the same characteristic and the same minimal polynomials.*

Solution. In one direction this is clear: similar matrices have the same characteristic and the same minimal polynomials.

Let A be a 3×3 matrix with minimal polynomial m_A and characteristic polynomial c_A. If $\deg m_A = 3$, then m_A is the only invariant factor of A. If $\deg m_A = 2$, then A has two invariant factors, $p_2 = m_A$ and $p_1 = c_A/m_A$ of degree 1. If $\deg m_A = 1$, then, since all other invariant factors of A must divide m_A, A has three invariant factors all equal to m_A. In any case the invariant factors of A are uniquely deined by m_A and c_A, so all 3×3 matrices with minimal polynomial m_A and characteristic polynomial c_A are similar.

5pt **12.2.10.** *Find all similarity classes of 6×6 matrices over $\mathbb{Q}$ with minimal polynomial $(x+2)^2(x-1)$.*

Solution. If A is a 6×6 matrix with the minimal polynomial $m_A = (x+2)^2(x-1)$, the other invariant factors of A must divide m_A and one the other, and have total degree 3; the only options are:
one more polynomial $(x+2)^2(x-1)$;
two more polynomials $(x+2)$ and $(x+2)(x-1)$;
two more polynomials $(x-1)$ and $(x+2)(x-1)$;
two more polynomials $(x+2)$ and $(x+2)^2$;
three more polynomials $(x+2)$, $(x+2)$, $(x+2)$;
three more polynomials $(x-1)$, $(x-1)$, $(x-1)$.
The similarity classes of such matrices correspond to these 6 distinct sets of invariant factors.

5pt **12.2.18.** *Let V be a finite dimensional vector space over $\mathbb{Q}$ and suppose T is a nonsingular (invertible) linear transformation of V such that $T^{-1} = T^2 + T$. Prove that $\dim V$ is divisible by 3 and that all such transformations of V are similar.*

Solution. Multiplying the identity $T^2 + T = T^{-1}$ by T we get that $T^3 + T^2 = I$, so $p(T) = 0$ for $p(x) = x^3 + x^2 - 1$. p is irreducible in $\mathbb{Q}[x]$ (otherwise it would have a root in $\mathbb{Z}$ equal to ± 1, which is not the case). Thus p is the minimal polynomial for T, and all other invariant factors of T must divide p and so, be equal to p. Hence, $\dim V = 3k$ where k is the nuber of invariant factors of T, and all such transformation of V have the same set of invariant factors and so, are similar.