15% **1.** *Let R be an integral domain, F be the field of fractions of R, and M be an R-module. Prove that $M/\operatorname{Tor}(M)$ is isomorphic to an R-submodule of an F-vector space.*

Solution. The product $V = F \otimes_R M$ is (by the extension of scalars $R \mapsto F$) an F-vector space, the R-module homomorphism $M \longrightarrow V$, $u \mapsto 1 \otimes u$, has kernel $\operatorname{Tor}(M)$, so $M/\operatorname{Tor}(M)$ is isomorphic to an R-submodule of V.

15% **2.** *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}^2$.*

(a) *Find the dimension of the $\mathbb{F}_2$-vector space $\mathbb{Z}_2 \otimes_{\mathbb{Z}} G$.*

Solution. Since for any $n, m \in \mathbb{N}$, $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z}_m \cong \mathbb{Z}_d$ where $d = \gcd(m, n)$ and $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}_n$, the $\mathbb{Z}$-module $\mathbb{Z}_2 \otimes_{\mathbb{Z}} G \cong (\mathbb{Z}_2 \otimes \mathbb{Z}_2) \oplus (\mathbb{Z}_2 \otimes \mathbb{Z}_3) \oplus (\mathbb{Z}_2 \otimes \mathbb{Z}_4) \oplus \mathbb{Z}_2 \otimes \mathbb{Z})^2 \cong \mathbb{Z}_2 \oplus 0 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2^2 \cong \mathbb{Z}_2^4$ has the structure of a 4-dimensional $\mathbb{F}_2$-vector space.

(b) *Find the dimension of the $\mathbb{F}_3$-vector space $\mathbb{Z}_3 \otimes_{\mathbb{Z}} G$.*

Solution. $\mathbb{Z}_3 \otimes_{\mathbb{Z}} G \cong 0 \oplus \mathbb{Z}_3 \oplus 0 \oplus \mathbb{Z}_3^2 \cong \mathbb{Z}_3^3$ is a 3-dimensional $\mathbb{F}_3$-vector space.

(c) *Find the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} G$.*

Solution. Since $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_n = 0$ for all n and $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Q}$, $\mathbb{Q} \otimes G \cong \mathbb{Q}^2$, which is a 2-dimensional $\mathbb{Q}$-vector space.

3. *Let M and N be free modules over an integral domain R with $\operatorname{rank} M = \operatorname{rank} N < \infty$ and let $\varphi\colon M \longrightarrow N$ be a homomorphism.*

10% (a) *If φ is surjective, prove that φ is an isomorphism.*

Solution. We have $\operatorname{rank}(\varphi(M)) = \operatorname{rank} M - \operatorname{rank}(\ker \varphi)$. If φ is surjective, then $\varphi(M) = N$, so $\operatorname{rank}(\varphi(M)) = \operatorname{rank} N = \operatorname{rank} M$, so $\operatorname{rank}(\ker \varphi) = 0$, so $\ker \varphi$ is a torsion module. But $\ker \varphi$ is a submodule of a torsion-free module M, so $\ker \varphi = 0$.

10% (b) *If φ is injective, prove that $N/\varphi(M)$ is a torsion module.*

Solution. If φ is injective, then $\operatorname{rank}(\varphi(M)) = \operatorname{rank} M = \operatorname{rank} N$, so $\operatorname{rank}(N/\varphi(M)) = 0$, so $N/\varphi(M)$ is a torsion module.

4. *Let R be a commutative unital ring, I be an ideal in R, and M be an R-module.*

10% (a) *Use the exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$ to prove that $(R/I) \otimes M \cong M/IM$.*

Solution. After tensor-multiplying this sequence by M, we get the exact sequence $I \otimes M \longrightarrow R \otimes M \longrightarrow (R/I) \otimes M \longrightarrow 0$. We have $R \otimes M \cong M$, $a \otimes u \mapsto au$ under this isomorphism, elements $a \otimes u$ of $I \otimes M$ are mapped to elements au of IM. Since IM is generated by such elements, the image of $I \otimes M$ in M is IM. Thus by the 1st isomorphism theorem, $M/IM \cong (R/I) \otimes M$.

10% (b) *If M is flat, prove that $I \otimes M \cong IM$.*

Solution. If M is flat, then the sequence $0 \longrightarrow I \otimes M \longrightarrow R \otimes M$ is exact, that is, the homomorphism $I \otimes M \longrightarrow R \otimes M \cong M$ is injective. Since the image of this homomorphism is IM, we get that $I \otimes M \cong IM$.

15% **5.** *Suppose M is a free module of finite rank over a PID and N is a submodule of M. Prove that N is a free summand of M (i.e. there exists a submodule N' of M such that $N \oplus N' = M$) iff the module M/N is torsion-free.*

Solution. If N is a free summand, $M = N \oplus N'$ for some N', then $M/N \cong N'$, and N' is torsion-free (as a submodule of torsion-free module M), so M/N is torsion free.

If M/N is torsion-free, then, since R is a PID, M/N is free, so projective, thus the exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ splits, and so N is a direct summand of M.

15% **6.** *Let V be an n-dimensional vector space, let T be a linear transformation $V \longrightarrow V$, and assume that the minimal polynomial m_T of T has degree n. Prove that the action of T on V is cyclic (V is a cyclic $F[x]$-module).*

Solution. The characteristic polynomial c_T of T has degree $\dim V = n$ and is divisible by m_T, so $c_T = m_T$. Also, c_T is the product of all invariant factors of T; hence, T has a single invariant factor m_T, and so, V is a cyclic $F[x]$-module.