Some integrals from Spivak

You may be surprised how carelessly we compute integrals: $\sqrt{1-\sin^2 x}$ is always assumed to be equal to $\cos x$, though this is only true when $\cos x \geq 0$; we derive from $x = \sin t$ that $t = \arcsin x$, though it is only true for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$; etc. This is a tradition; we try to get a formula for $\int f$ which would work for at least some x-s, then there is a good chance that this formula will give $\int f$ for all x-s. This may however lead to mistakes. For example, $\log x$ is primitive for $\frac{1}{x}$ only if $x > 0$; but $\frac{1}{x}$ is continuous on $(-\infty, 0)$ and so, must have another primitive on this interval. (This is $\log(-x)$.) See problem 10(ii) below for a more serious "paradox"...

My solutions and answers may be erroneous, please let me know if you find a mistake. Also note that some elementary functions can be written in different ways (for example, $\arcsin x = \frac{\pi}{2} - \arccos x$), and therefore, even correct answers may look differently.

1. (iii) $$\int \frac{e^x + e^{2x} + e^{3x}}{e^{4x}} dx = \int e^{-3x} dx + \int e^{-2x} dx + \int e^{-x} dx = -\frac{1}{3} e^{-3x} - \frac{1}{2} e^{-2x} - e^{-x} + C$$

(vii) $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{\frac{1}{a} dx}{\sqrt{1 - \frac{x^2}{a^2}}} = \int \frac{d(\frac{x}{a})}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} = \arcsin \frac{x}{a} + C$$

(By $df(x)$ I denote $f'(x)dx$, which is sometimes very convenient.)

(x) $$\int \frac{dx}{\sqrt{2x - x^2}} = \int \frac{dx}{\sqrt{1 - (x-1)^2}} = \int \frac{d(x-1)}{\sqrt{1 - (x-1)^2}} = \arcsin(x-1) + C$$

2. (ii) $$\int x e^{-x^2} dx = \frac{-1}{2} \int e^{-x^2} d(-x^2) = -\frac{1}{2} e^{-x^2} + C$$

(iv) $$\int \frac{e^x dx}{e^{2x} + 2e^x + 1} = \int \frac{de^x}{e^{2x} + 2e^x + 1} \underset{(u=e^x)}{=} \int \frac{du}{u^2 + 2u + 1} = \int \frac{d(u+1)}{(u+1)^2} = -\frac{1}{u+1} + C = \frac{-1}{e^x + 1} + C$$

(vi) $$\int \frac{x\, dx}{\sqrt{1 - x^4}} = \frac{1}{2} \int \frac{d(x^2)}{\sqrt{1 - x^4}} \underset{(u=x^2)}{=} \frac{1}{2} \int \frac{du}{\sqrt{1 - u^2}} = \frac{1}{2} \arcsin u + C = \frac{1}{2} \arcsin(x^2) + C$$

(viii) $$\int x \sqrt{1 - x^2} dx = \frac{-1}{2} \int \sqrt{1 - x^2} d(1 - x^2) = \frac{-1}{2} \cdot \frac{2}{3} (1 - x^2)^{3/2} + C = -\frac{1}{3} (1 - x^2)^{3/2} + C$$

(x) $$\int \frac{\log(\log x)}{x \log x} dx = \int \log(\log x) \frac{d \log x}{\log x} = \int \log(\log x) d \log(\log x) = \frac{1}{2} \big(\log(\log x)\big)^2 + C$$

3. (ii) $$\int x^3 e^{x^2} dx = \frac{1}{2} \int x^2 e^{x^2} d(x^2) \underset{(y=x^2)}{=} \frac{1}{2} \int y e^y dy = \frac{1}{2} \int y\, de^y \underset{\text{(by parts)}}{=} \frac{1}{2} \big(y e^y - \int e^y dy\big) = \frac{1}{2} (y e^y - e^y)$$
$$= \frac{1}{2} e^{x^2} (x^2 - 1) + C$$

(iv) $$\int x^2 \sin x\, dx = -\int x^2 d \cos x \underset{\text{(by parts)}}{=} -x^2 \cos x + \int \cos x\, d(x^2) = -x^2 \cos x + 2 \int x \cos x\, dx$$
$$= -x^2 \cos x + 2 \int x\, d \sin x \underset{\text{(by parts)}}{=} -x^2 \cos x + 2x \sin x - 2 \int \sin x\, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

(vi) $$\int \frac{\log(\log x)}{x} dx = \int \log(\log x) d \log x \underset{(u=\log x)}{=} \int \log u\, du \underset{\text{(by parts)}}{=} u \log u - \int u\, d \log u = u \log u - \int u \frac{1}{u} du$$
$$= u \log u - u + C = \log x \big(\log(\log x) - 1\big) + C$$

(x) $$\int x \log^2 x\, dx = \frac{1}{2} \int \log^2 x\, d(x^2) \underset{\text{(by parts)}}{=} \frac{1}{2} \big(x^2 \log^2 x - \int x^2 d(\log^2 x)\big) = \frac{1}{2} x^2 \log^2 x - \int x^2 \log x \frac{1}{x} dx$$
$$= \frac{1}{2} x^2 \log^2 x - \frac{1}{2} \int \log x\, d(x^2) \underset{\text{(by parts)}}{=} \frac{1}{2} x^2 \log^2 x - \frac{1}{2} \big(x^2 \log x - \int x^2 d \log x\big)$$
$$= \frac{1}{2} x^2 \log^2 x - \frac{1}{2} x^2 \log x + \frac{1}{2} \int x^2 \frac{1}{x} dx = \frac{1}{2} x^2 \log^2 x - \frac{1}{2} x^2 \log x + \frac{1}{4} x^2 + C$$

4. (ii) $$\int \frac{dx}{\sqrt{1+x^2}} \underset{(x=\tan u)}{=} \int \frac{d\tan u}{\sqrt{1+\tan^2 u}} = \int \frac{\frac{1}{\cos^2 u}du}{\frac{1}{\cos u}} = \int \frac{du}{\cos u}$$
$$= \log(\sec u + \tan u) + C = \log\bigl(\sqrt{1+\tan^2 u} + \tan u\bigr) + C = \log\bigl(\sqrt{1+x^2} + x\bigr) + C$$

(iv) $$\int \frac{dx}{x\sqrt{x^2-1}} \underset{\begin{pmatrix} y=\sqrt{x^2-1} \\ x=\sqrt{1+y^2} \\ dx=\frac{y\,dy}{\sqrt{1+y^2}} \end{pmatrix}}{=} \int \frac{y\,dy}{(1+y^2)y} = \arctan y + C = \arctan\sqrt{x^2-1} + C$$

Or:

$$\int \frac{dx}{x\sqrt{x^2-1}} \underset{\begin{pmatrix} x=\sec u \\ dx=\frac{\sin u\,du}{\cos^2 u} \end{pmatrix}}{=} \int \frac{\frac{\sin u\,du}{\cos^2 u}}{\frac{1}{\cos u}\sqrt{1-\sec^2 u}} = \int \frac{\frac{\sin u\,du}{\cos^2 u}}{\frac{1}{\cos u}\tan u} = \int u\,du = u + C = \sec^{-1} x + C$$

(Is this the same? Check.)

(vi) $$\int \frac{dx}{x\sqrt{1+x^2}} \underset{(x=\tan u)}{=} \int \frac{\frac{1}{\cos^2 u}du}{\tan u\sqrt{1+\tan^2 u}} = \int \frac{\frac{1}{\cos^2 u}du}{\frac{\sin u}{\cos^2 u}} = \int \frac{du}{\sin u} = -\log(\csc u + \cot u) + C$$
$$= -\log\bigl(\sqrt{1+\cot^2 u} + \cot u\bigr) + C = -\log\Bigl(\sqrt{1+\frac{1}{x^2}} + \frac{1}{x}\Bigr) + C = -\log\left(\frac{\sqrt{1+x^2}+1}{x}\right) + C$$

(viii) $$\int \sqrt{1-x^2}dx \underset{(x=\sin u)}{=} \int \sqrt{1-\sin^2 u}\,d\sin u = \int \cos^2 u\,du = \frac{1}{2}\int (1+\cos 2u)du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$$
$$= \frac{1}{2}(u + \sin u\cos u) + C = \frac{1}{2}\bigl(\arcsin x + x\sqrt{1-x^2}\bigr) + C$$

(x) $$\int \sqrt{x^2-1}\,dx \underset{(x=\sec u)}{=} \int \sqrt{\sec^2 u - 1}\,d\sec u = \int \tan u\frac{\sin u}{\cos^2 u}du = \int \frac{\sin^2 u}{\cos^3 u}du = \int \frac{\sin^2 u\,d\sin u}{\cos^4 u}$$
$$\underset{(y=\sin u)}{=} \int \frac{y^2 dy}{(1-y^2)^2} = \dots \quad \text{too difficult.}$$

Simpler:

$$\int \sqrt{x^2-1}\,dx \underset{(x=\cosh t)}{=} \int \sqrt{\cosh^2 t - 1}\,d\cosh t = \int \sinh^2 t\,dt = \frac{1}{2}\int (\cosh(2t)-1)dt = \frac{1}{2}\bigl(\frac{1}{2}\sinh(2t) - t\bigr) + C$$
$$= \frac{1}{2}(\cosh t\cdot\sinh t - t) + C = \frac{1}{2}\bigl(x\sqrt{x^2-1} - \cosh^{-1}x\bigr) + C = \frac{1}{2}\bigl(x\sqrt{x^2-1} - \log(x + \sqrt{x^2+1})\bigr) + C$$

5. (ii) $$\int \frac{dx}{1+e^x} \underset{\begin{pmatrix} u=e^x \\ x=\log u \\ dx=du/u \end{pmatrix}}{=} \int \frac{2u\,du}{u(1+u)} = \int \frac{du}{u} - \int \frac{du}{1+u} = \log u - \log(1+u) + C = x - \log(1+e^x) + C$$

(iv) $$\int \frac{dx}{\sqrt{1+e^x}} \underset{\begin{pmatrix} u=\sqrt{1+e^x} \\ du=\frac{e^x dx}{2\sqrt{1+e^x}} \\ dx=\frac{2u\,du}{u^2-1} \end{pmatrix}}{=} \int \frac{2u\,du}{u(u^2-1)} = 2\int \frac{du}{u^2-1} = \int \frac{du}{u-1} - \int \frac{du}{u+1}$$
$$= \log(u-1) - \log(u+1) + C = \log\frac{u-1}{u+1} + C = \log\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1} + C$$

(vi) $$\int \frac{dx}{\sqrt{\sqrt{x}+1}} \underset{\begin{pmatrix} y=\sqrt{\sqrt{x}+1} \\ x=(y^2-1)^2 \\ dx=4(y^2-1)dy \end{pmatrix}}{=} \int \frac{4(y^2-1)y\,dy}{y} = 4\int (y^2-1)dy = \frac{4}{3}y^3 - 4y + C = \frac{4}{3}y(y^2-3) + C$$
$$= \frac{4}{3}\sqrt{\sqrt{x}+1}(\sqrt{x}-2) + C$$

(viii) $$\int e^{\sqrt{x}}dx \underset{\begin{pmatrix} u=\sqrt{x} \\ x=u^2 \\ dx=2u\,du \end{pmatrix}}{=} \int e^u 2u\,du = 2\int u\,de^u \underset{\text{(by parts)}}{=} 2(ue^u - \int e^u du) = 2(ue^u - e^u) + C$$
$$= 2e^{\sqrt{x}}(\sqrt{x}-1) + C$$

6. (ii) $\displaystyle\int \frac{2x+1}{x^3-3x^2+3x-1}dx = \int \frac{2x+1}{(x-1)^3}dx \underset{\left(\substack{y=x-1\\x=y+1\\dx=dy}\right)}{=} \int \frac{2y+3}{y^3}dy = 2\int \frac{dy}{y^2}+3\int \frac{dy}{y^3} = -\frac{2}{y}-\frac{3}{2y^2}+C$

$$= -\frac{4y+3}{2y^2}+C = \frac{1-4x}{2(x-1)^2}+C$$

(iv) $\displaystyle\int \frac{2x^2+x+1}{(x+3)(x-1)^2}dx$. Writing $\displaystyle\frac{2x^2+x+1}{(x+3)(x-1)^2} = \frac{a}{x+3}+\frac{b}{x-1}+\frac{c}{(x-1)^2}$, find that $a=b=c=1$. So,

$$\int \frac{2x^2+x+1}{(x+3)(x-1)^2}dx = \int \frac{dx}{x+3}+\int \frac{dx}{x-1}+\int \frac{dx}{(x-1)^2} = \log(x+3)+\log(x-1)-\frac{1}{x-1}+C$$
$$= \log(x^2+2x-3)-\frac{1}{x-1}+C$$

(viii) $\displaystyle\int \frac{dx}{x^4+1}$. We have $x^4+1=(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$ and $\frac{1}{x^4+1} = \frac{1}{2\sqrt{2}}\left(\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}-\frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right)$.

$$\int \frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}dx = \int \frac{x+\frac{\sqrt{2}}{2}}{x^2+\sqrt{2}x+1}dx+\frac{\sqrt{2}}{2}\int \frac{dx}{x^2+\sqrt{2}x+1} = \frac{1}{2}\int \frac{d(x^2+\sqrt{2}x+1)}{x^2+\sqrt{2}x+1}+\int \frac{d(\sqrt{2}x+1)}{(\sqrt{2}x+1)^2+1}$$
$$= \frac{1}{2}\log(x^2+\sqrt{2}x+1)+\arctan(\sqrt{2}x+1)+C$$

$$\int \frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}dx = \int \frac{x-\frac{\sqrt{2}}{2}}{x^2-\sqrt{2}x+1}dx-\frac{\sqrt{2}}{2}\int \frac{dx}{x^2-\sqrt{2}x+1} = \frac{1}{2}\int \frac{d(x^2-\sqrt{2}x+1)}{x^2-\sqrt{2}x+1}-\int \frac{d(\sqrt{2}x-1)}{(\sqrt{2}x-1)^2+1}$$
$$= \frac{1}{2}\log(x^2-\sqrt{2}x+1)-\arctan(\sqrt{2}x-1)+C$$

So,

$$\int \frac{dx}{x^4+1} = \frac{1}{4\sqrt{2}}\log(x^2+\sqrt{2}x+1)+\frac{1}{2\sqrt{2}}\arctan(\sqrt{2}x+1)-\frac{1}{4\sqrt{2}}\log(x^2-\sqrt{2}x+1)+\frac{1}{2\sqrt{2}}\arctan(\sqrt{2}x-1)$$
$$= \frac{1}{4\sqrt{2}}\log\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}+\frac{1}{2\sqrt{2}}\arctan\frac{\sqrt{2}x}{1-x^2}+C$$

(I used the formula $\arctan u+\arctan v = \arctan\dfrac{u+v}{1-uv}$.)

8. (i) $\displaystyle\int \frac{\arctan x}{1+x^2}dx = \int \arctan x\, d\arctan x = \frac{1}{2}\arctan^2 x + C$

(iii) $\displaystyle\int \log\sqrt{1+x^2}dx \underset{\text{(by parts)}}{=} x\log\sqrt{1+x^2}-\int x\,d\log\sqrt{1+x^2}dx = x\log\sqrt{1+x^2}-\int x\frac{1}{\sqrt{1+x^2}}\cdot\frac{x\,dx}{\sqrt{1+x^2}}$

$$= x\log\sqrt{1+x^2}-\int \frac{x^2dx}{1+x^2} = x\log\sqrt{1+x^2}-\int dx+\int \frac{dx}{1+x^2} = x\log\sqrt{1+x^2}-x+\arctan x+C$$

(vi) $\displaystyle\int \arcsin\sqrt{x}dx \underset{\left(\substack{t=\arcsin\sqrt{x}\\x=\sin^2 t}\right)}{=} \int t\,d\sin^2 t \underset{\text{(by parts)}}{=} t\sin^2 t-\int \sin^2 t\,dt = t\sin^2 t-\int \frac{1-\cos 2t}{2}dt$

$$= t\sin^2 t-\frac{t}{2}+\frac{\sin 2t}{4}+C = t\big(\sin^2 t-\frac{1}{2}\big)+\frac{\sin t\cos t}{2}+C = \arcsin\sqrt{x}\cdot\big(x-\frac{1}{2}\big)+\frac{\sqrt{x}\sqrt{1-x}}{2}+C$$

(vii) $\displaystyle\int \frac{x\,dx}{1+\sin x} = \int \frac{x\,dx}{1+\cos(\frac{\pi}{2}-x)} = \int \frac{x\,dx}{2\cos^2(\frac{\pi}{4}-\frac{x}{2})} = -\int x\,d\tan(\frac{\pi}{4}-\frac{x}{2})$

$$\underset{\text{(by parts)}}{=} -x\tan(\frac{\pi}{4}-\frac{x}{2})+\int \tan(\frac{\pi}{4}-\frac{x}{2})dx \underset{\left(\substack{y=\frac{\pi}{4}-\frac{x}{2}\\dx=-2dy}\right)}{=} -x\tan(\frac{\pi}{4}-\frac{x}{2})-2\int \frac{\sin y}{\cos y}dy$$

$$= -x\tan(\frac{\pi}{4}-\frac{x}{2})+2\int \frac{d\cos y}{\cos y} = -x\tan(\frac{\pi}{4}-\frac{x}{2})+2\log\cos y+C = -x\tan(\frac{\pi}{4}-\frac{x}{2})+2\log\cos(\frac{\pi}{4}-\frac{x}{2})+C$$

$$\textbf{9.}\ \text{(i)}\quad \int \log(a^2+x^2)dx \underset{\text{(by parts)}}{=} x\log(a^2+x^2) - \int x\,d\log(a^2+x^2) = x\log(a^2+x^2) - \int \frac{2x^2dx}{a^2+x^2}$$
$$= x\log(a^2+x^2) - 2\int dx + 2a^2\int \frac{dx}{a^2+x^2} = x\log(a^2+x^2) - 2x + 2a\arctan\frac{x}{a} + C$$

(ii) $$\int \frac{1+\cos x}{\sin^2 x}dx = \int \frac{dx}{\sin^2 x} + \int \frac{d\sin x}{\sin^2 x} = -\cot x - \csc x + C$$

(iii) $\displaystyle\int \frac{x+1}{\sqrt{4-x^2}}dx.$ $\qquad \displaystyle\int \frac{dx}{\sqrt{4-x^2}} = \arcsin\frac{x}{2} + C$ by 1(vii)

$$\int \frac{x\,dx}{\sqrt{4-x^2}} = \frac{-1}{2}\int \frac{d(4-x^2)}{\sqrt{4-x^2}} = \frac{-1}{2}\cdot 2\sqrt{4-x^2} + C = -\sqrt{4-x^2} + C$$

So, $\displaystyle\int \frac{x+1}{\sqrt{4-x^2}}dx = \arcsin\frac{x}{2} - \sqrt{4-x^2} + C$

(iv) $$\int x\arctan x\,dx = \frac{1}{2}\int \arctan x\,d(x^2) \underset{\text{(by parts)}}{=} \frac{1}{2}\Big(x^2\arctan x - \int x^2 d\arctan x\Big)$$
$$= \frac{1}{2}\Big(x^2\arctan x - \int \frac{x^2dx}{1+x^2}\Big) = \frac{1}{2}\Big(x^2\arctan x - \int dx + \int \frac{dx}{1+x^2}\Big) = \frac{1}{2}\big(x^2\arctan x - x + \arctan x\big) + C$$

(v) $$\int \sin^3 x\,dx = -\int \sin^2 x\,d\cos x = -\int (1-\cos^2 x)d\cos x = \int \cos^2 x\,d\cos x - \int d\cos x$$
$$= \frac{1}{3}\cos^3 x - \cos x + C$$

$$\textbf{10.}\ \text{(i)}\quad \int \frac{dx}{(a^2+x^2)^2} = \frac{1}{a^4}\int \frac{dx}{(1+\frac{x^2}{a^2})^2} \underset{(y=\frac{x}{a})}{=} \frac{1}{a^3}\int \frac{dy}{(1+y^2)^2} \underset{\left(\substack{\text{reduction}\\ \text{formula 3}}\right)}{=} \frac{1}{a^3}\Big(\frac{1}{2}\cdot\frac{y}{1+y^2} + \frac{1}{2}\int \frac{dy}{1+y^2}\Big)$$
$$= \frac{1}{2a^3}\Big(\frac{y}{1+y^2} + \arctan y\Big) + C = \frac{1}{2a^2}\cdot\frac{x}{a^2+x^2} + \frac{1}{2a^3}\arctan\frac{x}{a} + C$$

(ii) $$\int \sqrt{1-\sin x}\,dx \underset{(y=\sin x)}{=} \int \sqrt{1-y}\,d\arcsin y = \int \sqrt{1-y}\frac{dy}{\sqrt{1-y^2}} = \int \frac{dy}{\sqrt{1+y}} = 2\sqrt{1+y} + C$$
$$= 2\sqrt{1+\sin x} + C$$

It's strange, isn't it? $\sqrt{1-\sin x}$ is a nonnegative continuous function, therefore its primitive must be increasing. But $2\sqrt{1+\sin x}$ is periodic and so, cannot be increasing. Can you find a mistake and correct the answer?

(iii) $$\int \arctan\sqrt{x}\,dx \underset{\text{(by parts)}}{=} x\arctan\sqrt{x} - \int x\,d\arctan\sqrt{x} = x\arctan\sqrt{x} - \int \frac{x\,d\sqrt{x}}{1+x}$$
$$\underset{(y=\sqrt{x})}{=} x\arctan\sqrt{x} - \int \frac{y^2dy}{1+y^2} = x\arctan\sqrt{x} - \int dy + \int \frac{dy}{1+y^2} = x\arctan\sqrt{x} - y + \arctan y + C$$
$$= (x+1)\arctan\sqrt{x} - \sqrt{x} + C$$

(iv) $$\int \sin\sqrt{x+1}\,dx \underset{\left(\substack{y=\sqrt{x+1}\\ x=y^2-1\\ dx=2y\,dy}\right)}{=} 2\int \sin y\cdot y\,dx = -2\int y\,d\cos y = -2y\cos y + 2\int \cos y\,dy$$
$$= -2y\cos y + 2\sin y + C = -2\sqrt{x+1}\cos\sqrt{x+1} + 2\sin\sqrt{x+1} + C$$

$$\textbf{12.}\ \text{(i)}\quad \int \frac{dx}{1+\sin x} \underset{(t=\tan\frac{x}{2})}{=} \int \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}} = 2\int \frac{dt}{(1+t)^2} = -\frac{2}{1+t} + C = -\frac{2}{1+\tan\frac{x}{2}} + C$$

(iii) $$\int \frac{dx}{a\sin x + b\cos x} \underset{(t=\tan\frac{x}{2})}{=} \int \frac{\frac{2dt}{1+t^2}}{a\frac{2t}{1+t^2} + b\frac{1-t^2}{1+t^2}} = 2\int \frac{dt}{b+2at-bt^2} = \frac{2}{b}\int \frac{dt}{1-t^2+2\frac{a}{b}t}$$
$$= \frac{2}{b}\int \frac{dt}{1+\frac{a^2}{b^2}-(t-\frac{a}{b})^2} = \frac{2}{b}\cdot\frac{1}{2\sqrt{1+\frac{a^2}{b^2}}}\Big(\int \frac{dt}{\sqrt{1+\frac{a^2}{b^2}}-(t-\frac{a}{b})} + \int \frac{dt}{\sqrt{1+\frac{a^2}{b^2}}+(t-\frac{a}{b})}\Big)$$
$$= \frac{1}{\sqrt{b^2+a^2}}\log\frac{\sqrt{a^2+b^2}+bt-a}{\sqrt{a^2+b^2}-bt+a} + C = \frac{1}{\sqrt{a^2+b^2}}\log\frac{\sqrt{a^2+b^2}+b\tan\frac{x}{2}-a}{\sqrt{a^2+b^2}-b\tan\frac{x}{2}+a} + C$$

(v) $$\int \frac{dx}{3+5\sin x} \underset{(t=\tan\frac{x}{2})}{=} \int \frac{\frac{2dt}{1+t^2}}{3+\frac{10t}{1+t^2}} = \frac{2}{3}\int \frac{dt}{t^2+\frac{10}{3}t+1} = \frac{2}{3}\int \frac{dt}{(t+\frac{5}{3})^2-\frac{16}{9}} = \frac{2}{3}\cdot\frac{1}{2\cdot\frac{4}{3}}\log\frac{t+\frac{5}{3}-\frac{4}{3}}{t+\frac{5}{3}+\frac{4}{3}}+C$$
$$= \frac{1}{4}\log\frac{3t+1}{3t+9}+C = \frac{1}{4}\log\frac{3\tan\frac{x}{2}+1}{3\tan\frac{x}{2}+9}+C$$

21. (a) $$\int \sin^n x\,dx = -\int \sin^{n-1} x\,d\cos x \underset{\text{(by parts)}}{=} -\sin^{n-1} x\cos x + \int \cos x\,d\sin^{n-1} x$$
$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,dx$$
$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$$

Thus $n\int \sin^n x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx$ and so,

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

(b) $$\int \cos^n x\,dx = \int \cos^{n-1} x\,d\sin x = \cos^{n-1} x\sin x - \int \sin x\,d\cos^{n-1} x$$
$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\sin^2 x\,dx = \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\,dx$$
$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

Thus $n\int \cos^n x\,dx = \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx$ and so,

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

(c) $$\int \frac{dx}{(x^2+1)^n} = \int \frac{dx}{(x^2+1)^{n-1}} - \int \frac{x^2dx}{(x^2+1)^n} = \int \frac{dx}{(x^2+1)^{n-1}} - \frac{1}{2}\int \frac{x\,d(x^2)}{(x^2+1)^n}$$
$$= \int \frac{dx}{(x^2+1)^{n-1}} + \frac{1}{2(n-1)}\int x\,d((x^2+1)^{-n+1})$$
$$\underset{\text{(by parts)}}{=} \int \frac{dx}{(x^2+1)^{n-1}} + \frac{1}{2n-2}\cdot\frac{x}{(x^2+1)^{n-1}} - \frac{1}{2n-2}\int \frac{dx}{(x^2+1)^{n-1}}$$
$$= \frac{1}{2n-2}\cdot\frac{x}{(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{dx}{(x^2+1)^{n-1}}$$