Try to figure out the Basel problem.
Note that

$$\int_0^1 x^n \mathrm{d}x = \frac{1}{n+1}$$

Then

$$\int_0^1 x^n \mathrm{d}x \int_0^1 y^n \mathrm{d}y = \int_0^1 \int_0^1 (xy)^n \mathrm{d}x\mathrm{d}y = \frac{1}{(n+1)^2}$$

So,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \int_0^1 \int_0^1 (xy)^{n-1} \mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^1 \frac{1}{1-xy} \mathrm{d}x\mathrm{d}y.$$

Given the change of variables

$$y = \frac{u+v}{\sqrt{2}}, \qquad x = \frac{u-v}{\sqrt{2}},$$

we compute the four partial derivatives:

$$x_u = \frac{1}{\sqrt{2}}, \qquad x_v = -\frac{1}{\sqrt{2}},$$

$$y_u = \frac{1}{\sqrt{2}}, \qquad y_v = \frac{1}{\sqrt{2}}.$$

The Jacobian determinant is

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix}.$$

Compute the determinant:

$$J = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{2} = 1.$$

Therefore,

$$\boxed{J = 1}.$$

then

$$\int_0^1 \int_0^1 \frac{1}{1-xy} \mathrm{d}x\mathrm{d}y = \int_{-\sqrt{2}/2}^{0} \int_{-v}^{\sqrt{2}+v} \frac{1}{1-\frac{u^2-v^2}{2}} \mathrm{d}u\mathrm{d}v + \int_0^{\sqrt{2}/2} \int_v^{\sqrt{2}-v} \frac{1}{1-\frac{u^2-v^2}{2}} \mathrm{d}u\mathrm{d}v.$$

This equals

$$2\int_0^{\sqrt{2}/2} \int_v^{\sqrt{2}-v} \frac{1}{1+\frac{v^2}{2}-\frac{u^2}{2}} \mathrm{d}u\mathrm{d}v = 2\int_0^{\sqrt{2}/2} \int_0^{u} \frac{1}{1+\frac{v^2}{2}-\frac{u^2}{2}} \mathrm{d}v\mathrm{d}u + 2\int_{\sqrt{2}/2}^{\sqrt{2}} \int_0^{\sqrt{2}-u} \frac{1}{1+\frac{v^2}{2}-\frac{u^2}{2}} \mathrm{d}v\mathrm{d}u \equiv 2A + 2B.$$

Then,

$$A = \int_0^{\sqrt{2}/2} \frac{2}{\sqrt{2-u^2}} \arctan\left(\frac{u}{\sqrt{2-u^2}}\right) \mathrm{d}u = \int_0^{\sqrt{2}/2} \frac{\sqrt{2}}{\sqrt{1-\left(\frac{u}{\sqrt{2}}\right)^2}} \arctan\left(\frac{\frac{u}{\sqrt{2}}}{\sqrt{1-\left(\frac{u}{\sqrt{2}}\right)^2}}\right) \mathrm{d}u$$

Similarly,

$$B = \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{1-\left(\frac{u}{\sqrt{2}}\right)^2}} \arctan\left(\frac{1-\frac{u}{\sqrt{2}}}{\sqrt{1-\left(\frac{u}{\sqrt{2}}\right)^2}}\right) \mathrm{d}u.$$

Let

$$u = \sqrt{2}\sin\theta, \qquad \theta \in [0, \tfrac{\pi}{6}].$$

Then

$$A=\int_0^{\pi/6}\frac{\sqrt{2}}{\sqrt{1-(\sin\theta)^2}}\ \arctan\left(\frac{\sin\theta}{\sqrt{1-(\sin\theta)^2}}\right)\sqrt{2}\cos\theta\mathrm{d}\theta=\int_0^{\pi/6}2\theta\mathrm{d}\theta=\frac{\pi^2}{36}$$

Also, we have

$$B=\int_{\pi/6}^{\pi/2}\frac{\sqrt{2}}{\sqrt{1-(\sin\theta)^2}}\ \arctan\left(\frac{1-\sin\theta}{\sqrt{1-(\sin\theta)^2}}\right)\sqrt{2}\cos\theta\mathrm{d}\theta=\int_{\pi/6}^{\pi/2}2\arctan\left(\frac{1-\sin\theta}{\cos\theta}\right)\mathrm{d}\theta$$

Where

$$\frac{1-\sin(\theta)}{\cos\theta}=\left(1-\frac{2\tan(\theta/2)}{1+\big(\tan(\theta/2)\big)^2}\right)\Bigg/\left(\frac{1-\big(\tan(\theta/2)\big)^2}{1+\big(\tan(\theta/2)\big)^2}\right)=\frac{1-2\tan(\theta/2)+\big(\tan(\theta/2)\big)^2}{1-\big(\tan(\theta/2)\big)^2}=\frac{1-\tan(\theta/2)}{1+\tan(\theta/2)}$$

Moreover,

$$\frac{1-\tan(\theta/2)}{1+\tan(\theta/2)}=\frac{\tan(\pi/4)-\tan(\theta/2)}{1+\tan(\pi/4)\tan(\theta/2)}=\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

Thus,

$$B=\int_{\pi/6}^{\pi/2}2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\mathrm{d}\theta=\int_{\pi/6}^{\pi/2}\left(\frac{\pi}{2}-\theta\right)\mathrm{d}\theta=\frac{\pi^2}{6}-\frac{1}{2}\left(\frac{\pi^2}{4}-\frac{\pi^2}{36}\right)=\frac{\pi^2}{18}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\int_0^1\int_0^1\frac{1}{1-xy}\mathrm{d}x\mathrm{d}y=2A+2B=2\left(\frac{\pi^2}{36}+\frac{\pi^2}{18}\right)=\frac{\pi^2}{6}$$