Real analysis

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0. Prerequisites

0.1. Disclaimer

Real analysis is an area of *pure*, or *theoretical* mathematics; this means that our main concern will be *proving* facts. Pure mathematics deals with abstract, ideal objects that exist only in our minds, and the only thing we can do with them is talk about them. Since mathematics is a science, we must be able to determine what is true and what is not; the only way to figure it out is by using *logic*. However, our everyday logic is imperfect, it is full of contradictions and paradoxes. Because of this, mathematicians have elaborated some special contradiction-free logic.

Now, any *math theory* is constructed as follows: one introduces some objects, operations and relations between them, and describes their properties, called *postulates* or *axioms*. Then, applying math logic, one deduces corollaries of the axioms, called *theorems*, *lemmas*, or *propositions*. This process is called *proving*; no obvious fact is considered true until it is perfectly proven, which makes mathematics "the art of proving". One introduces new objects into this game by *defining* them in such a way that no misinterpretaion may arise. Now, any statement related to the theory can be proved to be true or false (or neither!), but never both.

This turns mathematics into a game, like chess, with precise rules. However, these rules are not easy to learn, and following them directly would make mathematical texts a nightmare. Instead, they are learned through examples and used rather intuitively: you have to feel what counts as a rigorous proof, and what details can be skipped. So you are invited to play a game with strict rules, which rules are not explicitly explained to you! (This is unfair; it is no wonder that you can get confused and lost. And my advice is: play this game only if you like it, otherwise don't take this course; unless you plan to become a professional mathematician, the chance that you will actually need it some day is negligible.)

0.2. Elements of logic

Although we don't study math logic in this course, we'll use some logical notations. Logic operates with statements, which may be true or false (or neither); the logical operations are:

(i) $\wedge$, & – "and": the statement $A \wedge B$ is true if and only if both statements A and B are true.

(ii) $\vee$ – "or": $A \vee B$ is true if and only if at least one of A, B is true, including the case where both are true. ("or" is always *inclusive* in math.)

(iii) $\neg$ – "not": $\neg A$ is true if and only if A is false.

(iv) $\Rightarrow$ – implication. $A \Rightarrow B$ reads as "A implies B", "if A then B", "B follows from A", "A is sufficient for B", or "B is necessary for A".

(v) $\Leftrightarrow$ – equivalence, biconditional; it is equivalent to $(A \Rightarrow B) \wedge (B \Rightarrow A)$. $A \Leftrightarrow B$ reads as "A if and only if B", "A iff B", "A and B are equivalent", "A is necessary and sufficient for B".

There are also two *quantifiers*:

(i) The *universal* quantifier $\forall$ – "for all", "all", "any", "each", "every": the expression $(\forall x \in X)P(x)$, or just $(\forall x)P(x)$, says that a statement $P(x)$ that depends on x is true for all elements x of a set X.

(ii) The *existential* quantifier $\exists$ – "for some", "some", "exists", "there exists": the expression $(\exists x \in X)P(x)$, or just $(\exists x)P(x)$, says that a statement $P(x)$, which depends on x, is true for at least one element x of set X.

The correct usage of quantifiers is extremely important, a small change of quantifiers or even their order dramatically changes the meaning of the expression: "$\forall x \exists y ...$" is not the same as "$\exists y \forall x ...$". It is however customary in math texts to express quantifiers in words.

Also, the notation $\exists!$ is used sometimes, which reads as "there exists unique". (Notice, by the way, that $(\exists! x \in X)P(x)$ is equivalent to $(\exists x \in X)(P(x) \wedge (\forall y \in X)(P(y) \Rightarrow (y = x)))$.

The sign "=" – "equal to" – manifests the identity of two objects: "$A = B$" means that "A" and "B" are just two distinct notations for the same object, so that the symbols A and B are interchangeable.

0.3. Set theoretical notations and operations

Most math theories are based on the *set theory*, but we don't study it either in this course. We will however need some set-theoretical notations. We cannot define "sets" mathematically here; under "a set" we will understand "a collection of elements of any nature". (Which is not a definition, of course. And, actually, this approach, called *the naive set theory*, leads to contradictions!)

To express that a is an element of a set A we write $a \in A$; we also say that *a belongs to A* or *a is contained in A*. Every set is defined by its elements; we can express this by writing $\big(\forall a(a \in A \Longleftrightarrow a \in B)\big) \Rightarrow (A = B)$. (If for every object a, a is an element of A iff a is an element of B, then A and B are the same set. Or: iff A and B have the same elements, then $A = B$.)

To express that a set A has, say, elements a, b, c and no other elements, we write $A = \{a, b, c\}$. *The empty set* is the set with no elements; it is denoted by $\emptyset$. (So, $\emptyset = \{\}$.)

We say that B is *a subset of* A and write $B \subseteq A$ if every element of B is an element of A: $\forall a\big((a \in B) \Rightarrow (a \in A)\big)$, or $(\forall a \in B)(a \in A)$.

Let $P = P(a)$ be a statement that depends on elements a of a set A; then $\{a \in A \mid P(a)\}$ (or $\{a \in A : P(a)\}$ or $\{a \in A \text{ s.t. } P(a)\}$) is the subset of A that consists of all elements a of A for which $P(a)$ is true. (For example, if $\mathbb{Z}$ is the set of integers, $\{n \in \mathbb{Z} \mid (\exists k \in \mathbb{Z})(n = 2k)\}$ is the set of even integers.)

Another way to construct new sets is as follows: if for each element a of a set A an object $F(a)$ is somehow defined, $\{F(a) \mid a \in A\}$ is the set of all objects of the form $F(a)$ where $a \in A$. (For example, the set of even integers can be written as $\{2k \mid k \in \mathbb{Z}\}$.)

Given two sets A and B, their *union* $A \cup B$ is the set that consists of (all) elements of A and (all) elements of B: $A \cup B = \{a \mid a \in A \vee a \in B\}$, or $a \in A \cup B \iff a \in A \vee a \in B$.

Given two sets A and B, their *intersection* $A \cap B$ is the set that consists of (all) elements that belong to both A and B: $A \cap B = \{a \mid a \in A \wedge a \in B\}$, or $a \in A \cap B \iff a \in A \wedge a \in B$.

Two sets A and B are said to be *disjoint* if $A \cap B = \emptyset$, that is, if A and B have no common elements.

The union and the intersection are actually defined for arbitrary collection of sets (sets of sets): if $\mathcal{A}$ is a set whose elements are sets, then $\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$ is the set $\{x \mid (\exists A \in \mathcal{A}) x \in A\} = \{x \mid x \in A \text{ for some } A \in \mathcal{A}\}$ (the set of all elements that belong to at least one set from $\mathcal{A}$) and $\bigcap \mathcal{A} = \bigcap_{A \in \mathcal{A}} A$ is the set $\{x \mid (\forall A \in \mathcal{A}) x \in A\} = \{x \mid x \in A \text{ for all } A \in \mathcal{A}\}$ (the set of all elements that belong to all sets from $\mathcal{A}$). If $\mathcal{A}$ is a finite collection of sets, $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$, we also write $\bigcup_{A \in \mathcal{A}} A = \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$ and $\bigcap_{A \in \mathcal{A}} A = \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$.

Given two sets A and B, their *difference* $A \setminus B$ is the set whose elements are elements of A which are not elements of B: $A \setminus B = \{a \in A \mid a \notin B\}$.

Given two elements a and b, (a, b) denotes *the ordered pair* whose first element is a and the second element is b, so that $(a, b) = (c, d)$ iff $a = c$ and $b = d$. (And so, $(a, b) \neq (b, a)$ unless $a = b$.) Now, given two sets A and B, *the Cartesian product* $A \times B$ is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$: $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$. The Cartesian product $A_1 \times \cdots \times A_n$ of n sets $A_1, \ldots, A_n$ is defined inductively, by $A_1 \times \cdots \times A_n \times A_{n+1} = (A_1 \times \cdots A_n) \times A_{n+1}$, or as the set $\{(a_1, \ldots, a_n) \mid a_1 \in A_1, \ldots, a_n \in A_n\}$ of ordered n-tuples. We also write A^2 for $A \times A$, and A^n for the product $A \times \cdots \times A$ of n copies of A.

0.4. Mappings

Given two sets X and Y, *a relation* between X and Y is a subset R of the Cartesian product $X \times Y$; instead of $(x, y) \in R$ we write $x\,R\,y$. (An example is the order relation "$<$" on $\mathbb{R}$.)

A function, *a mapping*, or *a map* $f \colon X \longrightarrow Y$ is a relation between X and Y (that is, a subset of $X \times Y$) such that for every $x \in X$ there exists a unique $y \in Y$ such that $(x, y) \in f$. Instead of $(x, y) \in f$ we write $y = f(x)$; y is said to be *the image* of x under f or *the value* of f at x. X is called *the domain* of f and is denoted by $\mathrm{Dom}(f)$, Y is called *codomain* of f, the set $f(X) = \{f(x) \mid x \in X\}$ is called *the range* of f and is denoted by $\mathrm{Rng}(f)$. For a subset A of X, the subset $f(A) = \{f(x) \mid x \in A\}$ of Y is called *the image* of A; for a subset B of Y, the set $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$ of X is called *the inverse image*, or *the preimage* of B. For $A \subseteq X$, the function $g \colon A \longrightarrow Y$ defined by $g(x) = f(x)$, $x \in A$, is called *the restriction* of f to A and is denoted by $f|_A$; in this situation f is called *an extension* of g to X.

The graph of a function $f \colon X \longrightarrow Y$ is the set of points $(x, y) \in X \times Y$ such that $y = f(x)$; that is, by definition, the graph of f is f itslef, namely, the subset $\{(x, f(x)) \mid x \in A\}$ of $X \times Y$. In the case X and Y are subsets of the set $\mathbb{R}$ of real numbers (which we will introduce soon) the function $f \colon X \longrightarrow Y$ (which is the same as its graph) is a subset of *the Cartesian plane* $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

For a set X, the mapping $f \colon X \longrightarrow X$ defined by $f(x) = x$ for all $x \in X$ is called *the identity mapping* and is denoted by Id_X.

A function $f \colon X \longrightarrow Y$ is said to be *constant* if there is an element $y \in Y$ such that $f(x) = y$ for all $x \in X$. We write $f \equiv y$ in this case, or just $f = y$. And if we don't identify y, we write $f = \text{const}$.

For a set X, a mapping $X \times X \longrightarrow X$ is called *a binary operation* on X. (Addition and multiplication are binary operations on $\mathbb{R}$.)

A sequence $(x_n) = (x_1, x_2, \ldots)$ in a set X is a mapping $f \colon \mathbb{N} \longrightarrow X$, where $x_n \in X$ stands for $f(n)$, $n \in \mathbb{N}$.

Given two mappings $f \colon X \longrightarrow Y$ and $g \colon Y \longrightarrow Z$, *the composition* $g \circ f$ of f and g is the function $X \longrightarrow Z$ defined by $(g \circ f)(x) = g(f(x))$.

A mapping $f \colon X \longrightarrow Y$ is said to be *injective*, or *one-to-one*, if distinct elements of X have distinct images under f: for any $x, z \in X$, if $x \neq z$ then $f(x) \neq f(z)$. f is said to be *surjective*, or *onto*, if every element $y \in Y$ is the image of an element $x \in X$: for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$. f is said to be *bijective*, or a *one-to-one correspondence*, if it is both injective and surjective.

Given a mapping $f: X \longrightarrow Y$, a mapping $g: Z \longrightarrow X$, for $Z \subseteq Y$, is called *the inverse* of f and is denoted by f^{-1} if for any $x \in X$ and $y \in Z$, $f(x) = y$ iff $g(y) = x$. (Which implies that $Z = \mathrm{Dom}(g) = \mathrm{Rng}(f)$ and $X = \mathrm{Dom}(f) = \mathrm{Rng}(g)$.) Equivalently, $g = f^{-1}$ if $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Z$, that is, if $g(f(x)) = x$ for all $x \in X$ and $f(g(y)) = y$ for all $y \in Z$. If a mapping has an inverse it is said to be *invertible*; it is easy to see that a mapping is invertible iff it is injective, and so, is a bijection between X and $\mathrm{Rng}(f)$.

0.5. Types of proofs

Below is a list of standard types of proofs; it is not a complete classification of all types, and a single proof may combine elements of distinct types.

• *Direct proof* is guileless: using axioms and already proven facts you step-by-step approach and finally prove your theorem, – the statement you claim to be true.

• *Conditional proof* applies to conditional statement "$P \Rightarrow Q$", "if P then Q". In a theorem of this sort, P is called *the assumption* and Q *the conclusion* or *the assertion*; the theorem is usually formulated this way: "Assume that (or suppose that, or let) P, then Q". In a conditional proof, in addition to known facts, you assme that P is true (which doesn't have to be the case at all!), and deduce (prove) Q.

• *Biconditional proof* is a proof of a biconditional statement "$P \Leftrightarrow Q$". You can prove it in two ways: either you prove two conditional statements $P \Rightarrow Q$ and $Q \Rightarrow P$, or you connect P and Q by a chain of equivalent statements: you prove that $P \Leftrightarrow P_1$, $P_1 \Leftrightarrow P_2$, …, $P_k \Leftrightarrow Q$.

• *Proof by contraposition* of a conditional statement "$P \Rightarrow Q$" is a proof of the equivalent statement "$\neg Q \Rightarrow \neg P$": you assume that Q is false, and prove that P is false.

• Proving a biconditional statement $P \Leftrightarrow Q$ in two steps and using a proof by contraposition for one of them, you prove $P \Rightarrow Q$ and $\neg P \Rightarrow \neg Q$.

• *Proof by contradiction* of a statement P is made this way: you assume that P is false, based on this assumption come to any false statement, any contradiction, and conclude that P must be true. This is a very powerful type of proving, since you are not limited in the scope of contradictions you may get. But often in such a proof you just deduce P from $\neg P$: indeed, if $\neg P \Rightarrow P$, then it cannot be that $\neg P$ is true since in this case P is also true, which is a clear contradiciton.

• A special case of the proof by contradiction is that for conditional statements $P \Rightarrow Q$. In such a proof, you assume that $P \Rightarrow Q$ is false, that is, you assume $P \wedge \neg Q$, and deduce a contradiction. Often, the contradiction is reached by proving $\neg P$ or Q.

• If you are proving $P \wedge Q$, you may prove P and prove Q separately. If you are proving $R \Rightarrow P \wedge Q$, you may prove $R \Rightarrow P$ and $R \Rightarrow Q$ seperately.

• To prove $P \Rightarrow Q \vee R$, you may prove the equivalent statement $P \wedge \neg Q \Rightarrow R$ instead.

• To prove $P \wedge Q \Rightarrow R$, you may prove the equivalent statement $P \wedge \neg R \Rightarrow \neg Q$.

• The statement $P \vee Q \Rightarrow R$ can be proved by considering *cases*: you assume that P is true and prove R, then you assume that Q is true and prove R. You may introduce "cases" artificially: to prove a statement R you may prove $P \Rightarrow R$ and $\neg P \Rightarrow R$ for some statement P of your choice.

• If you need to prove $(\forall x \in X)P(x)$, that is, that $P(x)$ is true for all $x \in X$, you "fix" x (bound it, consider it a symbol that represents an element of X, but without specifying which element) and prove $P(x)$. Such a proof is usually starts with the words "Let x be an arbitrary element of X", or just "Let $x \in X$". (Of course, any other symbol can be used instead of x.)

• The statement $(\exists x \in X)P(x)$ can be proved *by example*: you just demonstrate an object x from X for which $P(x)$ is true. (This is not always possible, sometimes you have to prove the existence of such an element in different ways.)

• A special case of the above is when you need to disprove a statement $(\forall x \in X)P(x)$, that is, to prove $\neg\big((\forall x \in X)P(x)\big)$ (which is equivalent to $(\exists x \in X)\neg P(x)$). To have this done, it suffices to find $x \in X$ for which $P(x)$ is false. This method is called *proof by counterexample*.

• And finally, a warning. For a conditional statement $P \Rightarrow Q$, the statement $Q \Rightarrow P$ is called *the converse* of $P \Rightarrow Q$. The converse is not equivalent to the original statement! Don't try to prove $P \Rightarrow Q$ by proving $Q \Rightarrow P$, – it may be that the latter is true while the former is false.

It is a good style to make your proof short and optimal, avoiding statements or assumptions that are not really used in it. Thus, a direct proof, if exists, is preferrable over a proof by contradiction: in a direct conditional proof of $P \Rightarrow Q$ you only assume P, whereas in a proof of $P \Rightarrow Q$ by contradiction you assume both P and $\neg Q$.

1. Real numbers

1.1. Axioms

The main object real analysis deals with is the set of real numbers. Here is their definition: We have a nonempty set $\mathbb{R}$, whose elements are called *real numbers*. (Sometimes, we will also call real numbers *points*.) On $\mathbb{R}$, we have two operations: addition "$+$" and multiplication "$\cdot$" (which symbol is often just dropped): given two elements a and b of $\mathbb{R}$, an element of $\mathbb{R}$ is defined that is denoted by $a+b$ and called *the sum* of a and b, and an element $a \cdot b$, or just ab, of $\mathbb{R}$ is defined and called *the product* of a and b. Also, a subset P or $\mathbb{R}$, called *the set of positive real numbers*, is fixed. These two operations and the set P must have the following properties, that is, satisfy the following axioms:

(P1) $\forall a, b, c \in \mathbb{R}$, $(a+b)+c = a+(b+c)$ (the associativity law for addition).

(P2) $\forall a, b \in \mathbb{R}$, $a+b = b+a$ (the commutativity law for addition).

(P3) $\exists z \in \mathbb{R}$ such that $\forall a \in \mathbb{R}$ we have $a+z = a$ (existence of additive identity).

This element z is called *zero* and is denoted by 0.

(P4) $\forall a \in \mathbb{R}$ $\exists b \in \mathbb{R}$ such that $a+b=0$ (existence of additive inverses).

For every $a \in \mathbb{R}$ the corresponding element b is denoted by $-a$ and called *the additive inverse* of a.

(P5) $\forall a, b, c \in \mathbb{R}$, $(ab)c = a(bc)$ (the associativity law for multiplication).

(P6) $\forall a, b \in \mathbb{R}$, $ab = ba$ (the commutativity law for multiplication).

(P7) $\exists e \in \mathbb{R} \setminus \{0\}$ such that $\forall a \in \mathbb{R}$ we have $ae = a$ (existence of multiplicative identity).

This element e is called *one* and is denoted by 1.

(P8) $\forall a \in \mathbb{R} \setminus \{0\}$ $\exists b \in \mathbb{R}$ such that $ab = 1$ (existence of multiplicative inverses).

For every $a \in \mathbb{R}$ the corresponding element b is denoted by a^{-1} and called *the multiplicative inverse* or *the reciprocal* of a.

(P9) $\forall a, b, c \in \mathbb{R}$, $(a+b)c = ac+bc$ (the distributivity law).

(P10) $\forall a \in \mathbb{R}$ exactly one of the following is true: $a \in P$, $a = 0$, or $-a \in P$ (the trichotomy law).

If $a \in P$ we say that a is *positive*, if $-a \in P$ we say that a is *negative*.

(P11) $\forall a, b \in P$, $a+b \in P$ (closedness of P under addition).

(P12) $\forall a, b \in P$, $ab \in P$ (closedness of P under multiplication).

The last axiom (P13) is harder to state; it will be easier to do this after we give some definitions. We write "$b-a$" for "$b+(-a)$", we write "$a<b$" for "$b-a \in P$", and we write "$a \le b$" for "$(a<b) \vee (a=b)$". As we will see, using "$<$" we can reformulate (P10)-(P12) as follows:

(P10$'$) $\forall a, b \in \mathbb{R}$ exactly one of the following is true: $a<b$, $a=b$, or $b<a$.

(P10$''$) $\forall a, b, c \in \mathbb{R}$, if $a<b$ and $b<c$ then $a<c$.

(P11$'$) $\forall a, b, c \in \mathbb{R}$, if $a<b$ then $a+c<b+c$.

(P12$'$) $\forall a, b \in \mathbb{R}$ and $c \in P$, if $a<b$ then $ac<bc$.

(P10$'$) and (P10$''$) say that "$<$" is *an order relation* on $\mathbb{R}$; (P11$'$) and (P12$'$) say that this order relation is compatible with addition and multiplication.

For a set $A \subseteq \mathbb{R}$ and $b \in \mathbb{R}$ we say that b *is an upper bound of* A if $a \le b$ for all $a \in A$. We say that a set $A \subseteq \mathbb{R}$ is *bounded above* if there is an upper bound of A. We say that c is *the least* (or *minimal*) element of a set $B \subseteq \mathbb{R}$ if $c \in B$ and $c \le b$ for all $b \in B$. The least upper bound of a set A (if exists) is called *the supremum* of A and is denoted by $\sup A$. Now, the last axiom, called *the axiom of completeness*, can be stated as follows:

(P13) Every nonempty bounded above subset A of $\mathbb{R}$ has supremum. (In more details: for every nonempty $A \subseteq \mathbb{R}$, if A is bounded above, then there is c such that c is an upper bound of A and for every upper bound b of A we have that $c \le b$.)

Axioms (P1)-(P9) say that $\mathbb{R}$ is *a field*; axioms (P10)-(P12) (or rather (P10$'$)-(P12$'$)) add that this field is *ordered*; and axiom (P13) claims that this ordered field is *complete*. So, the set $\mathbb{R}$ of real numbers is defined as *a complete ordered field*. This is all we are assumed to know about real numbers; all other facts about them must be derived from these thirteen axioms.

A natural question is if this system of axioms is *consistent*, that is, not self-contradictory. It is consistent, and a possible way to prove this is *to construct* real numbers, that is, to construct a complete ordered field satisfying (P1)-(P13). Such a construction is briefly described in subsection 1.13 below.

1.2. Elementary properties of addition and multiplication

From (P1)-(P12) we will now derive more properties of real numbers, which we will prove and call *theorems*. The first our theorem is *the cancellation property* of addition:

Theorem 1.2.1. *For any $a, b, c \in \mathbb{R}$, if $a + c = b + c$, then $a = b$.*

Proof. The inverse $-c$ exists by (P4); let's add it to $a + c$, which is the same as $b + c$; we will then have $(a + c) + (-c) = (b + c) + (-c)$. But by (P1), $(a + c) + (-c) = a + (c + (-c)) = a + 0 = a$ and $(b + c) + (-c) = b + (c + (-c)) = b + 0 = b$, so $a = b$. ■

It follows (from commutativity of addition, (P2)) that the left-cancellation property also holds: if $c + a = c + b$, then $a = b$.

From now on I'll stop referring to the axioms I use.

Next, we prove that 0 is unique: there is only one element $b \in \mathbb{R}$ such that $a + b = a$ for all $a \in \mathbb{R}$, so "0" is uniquely defined. Moreover:

Theorem 1.2.2. *If $b \in \mathbb{R}$ is such that $a + b = a$ for some $a \in \mathbb{R}$, then $b = 0$.*

Proof. This easily follows from the cancellation property: since $a + b = a = a + 0$, we have $b = 0$. ■

The next theorem says that the additive inverse $-a$ of every element $a \in \mathbb{R}$ is also unique:

Theorem 1.2.3. *For any $a, b, c \in \mathbb{R}$, if $a + b = 0$ and $a + c = 0$ then $b = c$.*

Proof. $a + b = a + c$ so $b = c$ (by the left cancellation property). ■

Next,

Theorem 1.2.4. $-0 = 0$.

Proof. $0 + 0 = 0$, so $0 = -0$. ■

Theorem 1.2.5. *For any $a \in \mathbb{R}$, $-(-a) = a$.*

Proof. We have $(-a) + a = a + (-a) = 0$, so a is the additive inverse $-(-a)$ of $-a$. ■

Theorem 1.2.6. *For any $a, b \in \mathbb{R}$, $-(a + b) = (-a) + (-b)$.*

Proof. Indeed,

$$(a + b) + ((-a) + (-b)) = ((b + a) + (-a)) + (-b) = (b + (a + (-a)) + (-b) = (b + 0) + (-b) = b + (-b) = 0,$$

so $(-a) + (-b) = -(a + b)$. ■

For $a, b \in \mathbb{R}$ we define $a - b$ as $a + (-b)$.

Theorem 1.2.7. *For any $a, b \in \mathbb{R}$, $a = b$ iff $a - b = 0$.*

Proof. If $a = b$ then $a - b = a - a = a + (-a) = 0$. If $a - b = 0$, then $a = -(-b) = b$. ■

Theorem 1.2.8. *For any $a, b, c \in \mathbb{R}$, $(a + c) - (b + c) = a - b$.*

Proof.

$$\begin{aligned}(a + c) - (b + c) &= (a + c) + ((-b) + (-c)) = (a + c) + ((-c) + (-b)) = ((a + c) + (-c)) + (-b) \\ &= (a + (c + (-c))) + (-b) = (a + 0) + (-b) = a + (-b) = a - b.\end{aligned}$$

■

Theorem 1.2.9. *For any $a, b, c \in \mathbb{R}$, $(a - b) + (b - c) = a - c$.*

Proof.

$$
\begin{aligned}
(a-b)+(b-c) = (a+(-b))+(b+(-c)) = ((a+(-b))+b)+(-c) &= (a+((-b)+b))+(-c) \\
&= (a+0)+(-c) = a+(-c) = a-c.
\end{aligned}
$$

■

All these were properties of addition, in which proofs I only used axioms (P1)-(P4). (They say that $\mathbb{R}$ is *a commutative (or abelian) group* under addition.) But multiplication also satisfies similar axioms, (P5)-(P8)! (That is, $\mathbb{R} \setminus \{0\}$ with multiplication is also a commutative group.) So it also has similar properties, and there is no need to prove them! Ok, I'll reprove them – using copypaste and then replacing addition by multiplication, 0 by 1, and $-a$ by a^{-1}; we must only be careful with 0 and exclude it when needed:

Theorem 1.2.10. *For any $a, b \in \mathbb{R}$ and $c \in \mathbb{R} \setminus \{0\}$, if $ac = bc$, then $a = b$.*

Proof. The inverse c^{-1} exists by (P4), and we have $(ac)c^{-1} = (bc)c^{-1}$. But $(ac)c^{-1} = a(cc^{-1}) = a1 = a$ and $(bc)c^{-1} = b(cc^{-1}) = a1 = b$, so $a = b$. ■

It follows (from commutativity of multiplication) that the left-cancellation property also holds: if $ca = cb$ and $c \neq 0$, then $a = b$.

Next, we prove that 1 is unique:

Theorem 1.2.11. *If $b \in \mathbb{R}$ is such that $ab = a$ for some $a \in \mathbb{R} \setminus \{0\}$, then $b = 1$.*

Proof. This easily follows from the cancellation property: since $ab = a = a1$ and $a \neq 0$, we have $b = 1$. ■

The multiplicative inverse a^{-1} of every element $a \in \mathbb{R} \setminus \{0\}$ is also well defined:

Theorem 1.2.12. *For any $a \in \mathbb{R} \setminus \{0\}$ and $b, c \in \mathbb{R}$, if $ab = 1$ and $ac = 1$ then $b = c$.*

Proof. Since $ab = ac$ and $a \neq 0$, we have $b = c$. ■

Next,

Theorem 1.2.13. $1^{-1} = 1$.

Proof. $1 \cdot 1$, so $1 = 1^{-1}$. ■

Theorem 1.2.14. *For any $a \in \mathbb{R}$, $(a^{-1})^{-1} = a$.*

Proof. We have $a^{-1}a = aa^{-1} = 1$, so $a = (a^{-1})^{-1}$. ■

Theorem 1.2.15. *For any $a, b \in \mathbb{R} \setminus 0$, $(ab)^{-1} = a^{-1}b^{-1}$.*

Proof. Indeed,

$$
(ab)(a^{-1}b^{-1}) = (ba)(a^{-1}b^{-1}) = ((ba)a^{-1})b^{-1} = (b(aa^{-1}))b^{-1} = (b1)b^{-1} = bb^{-1} = 1,
$$

so $a^{-1}b^{-1} = (ab)^{-1}$. ■

For $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$ we define a/b, $a : b$, and $\frac{a}{b}$ as ab^{-1}.

Theorem 1.2.16. *For any $a, b \in \mathbb{R} \setminus \{0\}$, $a = b$ iff $a/b = 1$.*

Proof. If $a = b$ then $a/b = a/a = aa^{-1} = 1$. If $a/b = 1$, then $a = (b^{-1})^{-1} = b$. ■

We will now mix addition and multiplication:

Theorem 1.2.17. *For any $a \in \mathbb{R}$, $a0 = 0$.*

Proof. $a0 = a(0+0) = a0 + a0$, so $a0 = 0$. ■

A sort of converse of this theorem is also true:

Theorem 1.2.18. *For any $a, b \in \mathbb{R}$, if $ab = 0$ then $a = 0$ or $b = 0$.*

Proof. We can reformulate this statement this way: If $ab = 0$ and $a \neq 0$ then $b = 0$. In this form, it follows from the cancellation property of multiplication: since $ab = a0$ and $a \neq 0$, we have $b = 0$. ■

Next,

Theorem 1.2.19. *For any $a, b \in \mathbb{R}$, $(-a)b = -(ab)$, $(-a)(-b) = ab$, and $-a = (-1)a$.*

Proof. $ab + (-a)b = (a + (-a))b = 0b = 0$, so $(-a)b = -(ab)$.

$(-a)(-b) = -(a(-b)) = -(-(ab)) = ab$.

$(-1)a = -(1a) = -a$. ∎

1.3. Order

Now, let's discuss the order "$<$" on $\mathbb{R}$. Recall that we define "$a < b$" as "$b - a \in P$", and "$a \le b$" as "$(a = b) \vee (a < b)$". We also define "$a > b$" as "$b < a$", and "$a \ge b$" as "$b \le a$".

Theorem 1.3.1. *For any $a \in \mathbb{R}$, a is positive iff $a > 0$ and negative iff $a < 0$.*

Proof. "a is positive" means that $a \in P$; "$a > 0$" means that $a - 0 \in P$. Since $a - 0 = a$, these two statements are equivalent.

Finally, $a < 0$ iff $-a = 0 - a \in P$. ∎

Hence, we have the trichotomy: for any $a \in \mathbb{R}$ exactly one of the following is true: $a > 0$, $a = 0$, or $a < 0$. This can be strengthened:

Theorem 1.3.2. (P10$'$) *For any $a, b \in \mathbb{R}$ exactly one of the following is true: $a < b$, $a = b$, or $a > b$.*

Proof. We have $a < b$ iff $b - a \in P$, $a = b$ iff $b - a = 0$, and $a > b$ iff $a - b \in P$. Also, since $a - b = -(-a) + (-b) = -((-a) + b) = -(b - a)$, we have that $a > b$ iff $-(b - a) \in P$. Since exactly one of $b - a \in P$, $b - a = 0$, and $-(b - a) \in P$ holds, we obtain that exactly one of $a < b$, $a = b$, or $a > b$ holds. ∎

Also,

Theorem 1.3.3. (P10$''$) *For any $a, b, c \in \mathbb{R}$, if $a < b$ and $b < c$, then $a < c$.*

Proof. If $b - a \in P$ and $c - b \in P$, then $c - a = (c - b) + (b - a) \in P$. ∎

Theorems 1.3.2 and 1.3.3, (P10$'$) and (P10$''$), say that "$<$" is *an order relation* on $\mathbb{R}$: the elements of $\mathbb{R}$, real numbers, are *ordered*. The next two theorems say that this order "agrees" with addition and multiplication:

Theorem 1.3.4. (P11$'$) *For any $a, b, c \in \mathbb{R}$, if $a < b$, then $a + c < b + c$.*

Proof. Since $(b + c) - (a + c) = b - a$, we have $(b + c) - (a + c) \in P$ iff $b - a \in P$, so $a + c < b + c$ iff $a < b$. ∎

Theorem 1.3.5. (P12$'$) *For any $a, b, c \in \mathbb{R}$, if $a < b$ and $c > 0$, then $ac < bc$.*

Proof. Since $bc - ac = (b - a)c$, if $a < b$ and $c > 0$, that is, if $b - a, c \in P$, then $bc - ac \in P$, so $ac < bc$. ∎

We deduced theorems (P10$'$)-(P12$'$) from axioms (P10)-(P12) (along with (P1)-(P9), of course). Another approach is to introduce a relation "$<$" on $\mathbb{R}$ that satisfies (P10$'$)-(P12$'$), define $P = \{a \in \mathbb{R} \mid a > 0\}$, and prove (P10)-(P12) as theorems.

An element c of a subset A of $\mathbb{R}$ is called *the greatest* or *the maximal* element of A, and is denoted by $\max A$, if $a \le c$ for all $a \in A$. (Not every subset of $\mathbb{R}$ has a maximal element.) If $c \in A$ is such that $c \le a$ for all $a \in A$, c is called *the least* or *the minimal* element of A, and is denoted by $\min A$. (Not every subset of $\mathbb{R}$ has a minimal element.)

A Dedekind cut in $\mathbb{R}$ (or, actually, in any ordered set) is a pair (A, B) of nonempty subsets of $\mathbb{R}$ such that $A \cap B = \emptyset$, $A \cup B = \mathbb{R}$, and $a < b$ for all $a \in A$ and $b \in B$. An order is said to be *complete* if for every Dedekind cut (A, B), A has a maximal element or B has a minimal element; (P13) is equivalent to

(P13$'$) the order "$<$" on $\mathbb{R}$ is complete. (To deduce (P13$'$) from (P13), given a Dedekind cut (A, B), let $c = \sup A$; then $c \in A$ or $c \in B$. If $c \in A$, then c is the maximal element of A since it is an upper bound of A; if $c \in B$, then c is the minimal element of B since all elements of B are upper bounds of A and c is the least upper bound of A.)

Here are some more properties of "$<$":

Theorem 1.3.6. *For any $a, b \in \mathbb{R}$, if $b > 0$ then $a + b > a$ and $a - b < a$.*

Proof. We have $a + b > a + 0 = a$ since $b > 0$ and $a - b = a + (-b) < a + 0 = a$ since $-b < 0$. ∎

Theorem 1.3.7. *For any $a, b \in \mathbb{R}$, $a < b$ iff $-a > -b$.*

Proof. If $a < b$, then $-b = a + (-(a+b)) < b + (-(a+b)) = -a$. Conversely, if $-a > -b$, then $a = -(-a) < -(-b) = b$. ∎

Theorem 1.3.8. *For any $a, b, c \in \mathbb{R}$, if $a < b$ and $c < 0$, then $ac > bc$.*

Proof. We have $-c > 0$, so $-ac = a(-c) < b(-c) = -bc$, so $ac > bc$. ∎

Theorem 1.3.9. *Let $a \in \mathbb{R}$, $a > 0$. For any $b \in \mathbb{R}$, if $b > 1$, then $ab > a$; if $b < 1$, then $ab < a$.*

Proof. Since $a > 0$, if $b > 1$, then $ab = ba > 1a = a$. If $b < 1$, then $ab = ba < 1a = a$. ∎

Theorem 1.3.10. *For any $a, b \in \mathbb{R}$, if $a, b > 0$ then $ab > 0$; if $a > 0$ and $b < 0$, then $ab < 0$; if $a < 0$ and $b > 0$, then $ab < 0$; if $a, b < 0$, then $ab > 0$.*

Proof. If $a, b > 0$, then $ab > 0b = 0$. If $a > 0$, $b < 0$, then $ab = ba < 0a = 0$. The case $a < 0$, $b > 0$ is similar. If $a, b < 0$, then $-a, -b > 0$, so $ab = (-a)(-b) > 0$. ∎

Theorem 1.3.11. $1 > 0$.

Proof. (P7) says that $1 \neq 0$. If $1 < 0$, then $1 = 1 \cdot 1 > 0$, contradiction. So, $1 > 0$. ∎

We define $2 = 1 + 1$. Since $1 > 0$, we have $2 = 1 + 1 > 1 + 0 = 1$, and so $2 > 0$. We also define $3 = 2 + 1$ and $4 = 3 + 1$.

Theorem 1.3.12. *For any $a \in \mathbb{R} \setminus \{0\}$, $a > 0$ iff $a^{-1} > 0$.*

Proof. We have $aa^{-1} = 1 > 0$. So, a and a^{-1} are positive or negative simultaneously. ∎

1.4. Absolute value

For $a \in \mathbb{R}$ we define *the absolute value*, or *the modulus*, $|a|$ of a by $|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a < 0$.

Theorem 1.4.1. *For any $a \in \mathbb{R}$, $|a| \geq 0$ for all $a \in \mathbb{R}$, and $|a| = 0$ iff $a = 0$.*

Proof. If $a > 0$ then $|a| = a > 0$. If $a < 0$ then $|a| = -a > 0$. If $a = 0$ then $|a| = 0$. ∎

Theorem 1.4.2. *For any $a \in \mathbb{R}$, $|-a| = |a|$. Conversely, for any $a, b \in \mathbb{R}$, if $|a| = |b|$ then $a = b$ or $a = -b$.*

Proof. If $a > 0$ then $-a < 0$, so $|a| = a$ and $|-a| = -(-a) = a$. If $a < 0$ then $-a > 0$, so $|a| = -a$ and $|-a| = -a$. If $a = 0$ then $a = -a$ so $|a| = |-a|$.

Let $|a| = |b|$. If $a, b > 0$, then $a = |a| = |b| = b$. If $a > 0$ and $b < 0$, then $a = |a| = |b| = -b$. If $a < 0$ and $b > 0$, then $a = -|a| = -|b| = -b$. If $a, b < 0$, then $a = -|a| = -|b| = b$. ∎

Theorem 1.4.3. *For any $a \in \mathbb{R}$, $-|a| \leq a \leq |a|$.*

Proof. For any $a \neq 0$ we have $|a| > 0$ and $-|a| < 0$. If $a > 0$ then $|a| = a$ so $-|a| < 0 < a = |a|$. If $a < 0$ then $|a| = -a$, so $-|a| = a < 0 < a$. If $a = 0$ then $-|a| = a = |a|$.

Theorem 1.4.4. *For any $a, b \in \mathbb{R}$, $|a| < b$ iff $-b < a < b$, and $|a| \leq b$ iff $-b \leq a \leq b$.*

Proof. Let $|a| < b$. Since $|a| \geq 0$, we have $b > 0$ and $-b < 0$. If $a \geq 0$, then $|a| = a$, so $-b < 0 \leq a < b$. If $a < 0$, then $|a| = -a$, so $-b < 0 \leq -a < b$, so $b > a > -b$. (Or, simpler: $-b < -|a| \leq a \leq |a| < b$.)

Let $-b < a < b$. If $a \geq 0$, then $|a| = a < b$. If $a < 0$, then, since $-b < a$, we have $b > -a = |a|$.

Also, if $|a| = b$ then $-b \leq a = b$, and if $-b = a$ or $a = b$ then $|a| = b$.

Theorem 1.4.5. *For any $a, b \in \mathbb{R}$, $|ab| = |a| \cdot |b|$.*

(We say that absolute value is a *multiplicative* function.)

Proof. We have four cases.
If both $a, b > 0$ then also $ab > 0$, so $|ab| = ab = |a| \cdot |b|$.
If $a > 0$ and $b < 0$ then $ab < 0$, so $|ab| = -ab = a(-b) = |a| \cdot |b|$.
The case $a < 0$, $b > 0$ is similar.
If $a, b < 0$, then $ab > 0$, so $|ab| = ab = (-a)(-b) = |a| \cdot |b|$. ■

The following property of $|\cdot|$ is called *the triangle inequality*:

Theorem 1.4.6. *For any $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.*

Proof. Here we have six cases.
If both $a, b > 0$ then also $a + b > 0$, so $|a + b| = a + b = |a| + |b|$.
If $a > 0$, $b < 0$ and $a + b > 0$, then $|a + b| = a + b = |a| - |b| < |a| < |a| + |b|$ since $|b| > 0$.
If $a > 0$, $b < 0$ and $a + b < 0$, then $|a + b| = -(a + b) = (-a) + (-b) = -|a| + |b| < |b| < |a| + |b|$ since $|a| > 0$.
The (two) cases where $a < 0$ and $b > 0$ are similar.
Finally, if both $a, b < 0$ then $a + b < 0$ so $|a + b| = -(a + b) = (-a) + (-b) = |a| + |b|$. ■

Another proof. Since $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$ we have $-(|a| + |b|) = -|a| - |b| \leq a + b \leq |a| + |b|$, so $|a + b| \leq |a| + |b|$ by Theorem 1.4.4. ■

A more general version of the triangle inequality is

Theorem 1.4.7. *For any $a, b, c \in \mathbb{R}$, $|a - c| \leq |a - b| + |b - c|$.*

Proof. By Theorem 1.4.6, $|a - c| = \big|(a - b) + (b - c)\big| \leq |a - b| + |b - c|$. ■

The meaning of $|a - b|$ is that it is *the distance* between two "points" a and b. (Thus $|a| = |a - 0|$ is the distance between a and 0.) It is the triangle inequality that allows us to interprete $|\cdot|$ this way.

For two numbers a and b we define $\max\{a, b\} = a$ if $a \geq b$ and $= b$ if $a \leq b$, and $\min\{a, b\} = b$ if $a \geq b$ and $= a$ if $a \leq b$.

Theorem 1.4.8. *For any $a, b \in \mathbb{R}$, $\max\{a, b\} = \frac{a+b+|a-b|}{2}$ and $\min\{a, b\} = \frac{a+b-|a-b|}{2}$.*

Proof. W.l.o.g. (without loss of generality) we may assume that $a \geq b$. (This means that a and b are interchangable, so that to prove the case $b \geq a$ we simple replace a by b and b by a.) Then $a - b \geq 0$, so $\frac{a+b+|a-b|}{2} = (a + b + (a - b))/2 = (2a)2^{-1} = a = \max\{a, b\}$ and $\frac{a+b-|a-b|}{2} = (a + b - (a - b))/2 = (2b)2^{-1} = b = \min\{a, b\}$. ■

1.5. Squares and square roots

For $a \in \mathbb{R}$, we define $a^2 = aa$ and call it *the square of a*.

Theorem 1.5.1. *For any $a \in \mathbb{R}$, $a^2 \geq 0$, and $a^2 = 0$ iff $a = 0$.*

Proof. If $a = 0$ then $a^2 = 0$. If $a > 0$ then $a^2 = aa > 0$ (as a product of two positive numbers). If $a < 0$, then $a^2 = (-a)^2 > 0$ as $-a > 0$. ■

Theorem 1.5.2. *For any $a, b \in \mathbb{R}$, $a^2 = b^2$ iff $a = b$ or $a = -b$.*

[The proof is left to you.]

Theorem 1.5.3. *For any $a, b \geq 0$, $a^2 < b^2$ iff $a < b$.*

[The proof is left to you.]

Theorem 1.5.4. *For any $a, b \in \mathbb{R}$, $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$.*

Proof. $(a + b)^2 = (a + b)(a + b) = a(a + b) + b(a + b) = aa + ab + ba + bb = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$.
$(a - b)^2 = (a + (-b))^2 = a^2 + 2a(-b) + (-b)^2 = a^2 - 2ab + b^2$. ■

Now let $a, b, c \in \mathbb{R}$ with $a > 0$, and for $x \in \mathbb{R}$ let's consider the expression $ax^2 + bx + c$. We have

$$ax^2 + bx + c = a\big(x^2 + 2\tfrac{b}{2a}x + \big(\tfrac{b}{2a}\big)^2\big) - a\big(\tfrac{b}{2a}\big)^2 + c = \big(x + \tfrac{b}{2a}\big)^2 + c - \tfrac{b^2}{4a}. \tag{1.1}$$

Since $\big(x + \frac{b}{2a}\big)^2 \geq 0$, we obtain the following theorem:

Theorem 1.5.5. *Let $a, b, c \in \mathbb{R}$, $a > 0$. If $b^2 - 4ac < 0$, then $ax^2 + bx + c > 0$ for all $x \in \mathbb{R}$. If $b^2 - 4ac \geq 0$, then $ax^2 + bx + c \leq 0$ for some $x \in \mathbb{R}$.*

The number $b^2 - 4ac$ is called *the discriminant* of *the quadratic polynomial $ax^2 + bx + c$.*

Proof. Since $a > 0$ we have $b^2 - 4ac < 0$ iff $c - \frac{b^2}{4a} > 0$, and in this case the right hand part of (1.1) is positive for all x. If $b^2 - 4ac \leq 0$, then $c - \frac{b^2}{4a} \leq 0$, so for $x = -b/2a$ we have $ax^2 + bx + c = c - \frac{b^2}{4a} \leq 0$. ∎

Next, for $a \in \mathbb{R}$, we define *the square root* $\sqrt{a}$ of a as $b \geq 0$ such that $b^2 = a$. It follows from Theorem 1.5.1 that $\sqrt{a}$ may only exist if $a \geq 0$. It follows from Theorem 1.5.2 that $\sqrt{a}$, if exists, is unique. (There can be at most one $b > 0$ such that $b^2 = a$: if also $c^2 = a$, then either $c = b$ or $c = -b < 0$.)

We thus have:

Theorem 1.5.6. *For any $a \in \mathbb{R}$, $\sqrt{a^2} = |a|$ and for any $a \geq 0$, $(\sqrt{a})^2 = a$.*

Thanks to the axiom of completeness (P13), we can prove that $\sqrt{a}$ exists for all positive a:

Theorem 1.5.7. *For any $a > 0$, $\sqrt{a}$ exists.*

Proof. Let $a > 0$; consider the set

$$A = \{x \in \mathbb{R} \mid x > 0 \wedge x^2 \leq a\}.$$

A is nonempty: if $a \geq 1$ then $1 \in A$; if $a < 1$ then $a^2 < a$, so $a \in A$. For any $b > 0$, if $b^2 \geq a$ then b is an upper bound of A, since $x^2 \leq a \leq b^2$ and so $x \leq b$ for all $x \in A$. Hence, A is bounded above: if $a \geq 1$ then $a^2 > a$ so a is an upper bound of A; if $a < 1$ then $1^2 > a$ so 1 is an upper bound of A.

By (P13), A has supremum (the least upper bound); let $b = \sup A$, then $b > 0$. I claim that $b^2 = a$. We'll prove this *by contradiction*: let's assume that $b^2 \neq a$ and show that b is not the supremum of A in this case.

Assume that $b^2 < a$. Let $\varepsilon \in \mathbb{R}$ be such that $0 < \varepsilon < 1$. (ε is used to denote "a small number".) Then $(b + \varepsilon)^2 = b^2 + 2b\varepsilon + \varepsilon^2 < b^2 + 2b\varepsilon + \varepsilon$. Now if $\varepsilon < \frac{a-b^2}{2b+1}$ (for instance, we can take $\varepsilon = \frac{a-b^2}{2(2b+1)}$), then $b^2 + 2b\varepsilon + \varepsilon < b^2 + (a - b^2) = a$, so $b + \varepsilon \in A$. Since $b + \varepsilon > b$, this means that b is not an upper bound of A.

Assume that $b^2 > a$. Let $\varepsilon > 0$. Then $(b - \varepsilon)^2 = b^2 - 2b\varepsilon + \varepsilon^2 > b^2 - 2b\varepsilon$. Now if $\varepsilon < \frac{b^2-a}{2b}$ (for instance, we can take $\varepsilon = \frac{b^2-a}{4b}$), then $b^2 - 2b\varepsilon > b^2 - (b^2 - a) = a$, so $b - \varepsilon$ is an upper bound of A. Since $b - \varepsilon < b$, this means that b is not the least upper bound of A.

Hence, it cannot be that $b^2 < a$ or $b^2 > a$; so, $b^2 = a$. ∎

1.6. The arithmetic-geometric mean, the triangle, and the Cauchy-Schwarz inequalities

The fact that a^2 is always nonnegative helps prove various inequalities.

Theorem 1.6.1. *For any $a, b \in \mathbb{R}$ with $a \leq b$ we have $a \leq \frac{a+b}{2} \leq b$, and if $a \geq 0$, then also $a \leq \sqrt{ab} \leq \frac{a+b}{2} \leq b$. If a is strictly less than b, $a < b$, then all all the inequalities are strict.*

$\frac{a+b}{2}$ is called *the arithmetic mean* of a, b, $\sqrt{ab}$ *the geometric mean* of a, b, and $\sqrt{ab} \leq \frac{a+b}{2}$ *the arithemetic-geometric mean ineqiality.*

Proof. If $a = b$ then $a = \frac{a+b}{2} = b$, and if $a \geq 0$ then also $\sqrt{ab} = a$.

Suppose that $a < b$. Then $2a = a + a < a + b$, so $a < \frac{a+b}{2}$; and $a + b < b + b = 2b$, so $\frac{a+b}{2} < b$. If $0 \leq a < b$, then also $a^2 < ab$, so $a = \sqrt{a^2} < \sqrt{ab}$; and $ab < b^2$, so $\sqrt{ab} < \sqrt{b^2} = b$. Finally, if $0 \leq a < b$,

$$\tfrac{a+b}{2} - \sqrt{ab} = \tfrac{1}{2}\big((\sqrt{a})^2 + (\sqrt{b})^2 - 2\sqrt{a}\sqrt{b}\big) = \tfrac{1}{2}\big(\sqrt{a} - \sqrt{b}\big)^2 > 0,$$

so $\frac{a+b}{2} > \sqrt{ab}$. ∎

We can use "the theory of quadratic expressions", Theorem 1.5.5, to prove (the simplest version of) the fundamental *Cauchy-Schwarz inequality*:

Theorem 1.6.2. *For any $a_1, a_2, b_1, b_2 \in \mathbb{R}$,*

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) \geq (a_1 b_1 + a_2 b_2)^2 \tag{1.2}$$

(equivalently, $\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2} \geq |a_1 b_1 + a_2 b_2|$), where the equality holds iff $a_1 = a_2 = 0$ or there is $x \in \mathbb{R}$ such that both $b_1 = xa_1$ and $b_2 = xa_2$.

Proof. The proof is tricky. First of all, if $a_1 = a_2 = 0$ then both parts of (1.2) are equal to 0, and if there is an x such that $b_1 = xa_1$ and $b_2 = xa_2$, then

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) = x^2(a_1^2 + a_2^2)^2 = (a_1 x a_1 + a_2 x a_2)^2 = (a_1 b_1 + a_2 b_2)^2.$$

Assume that at least one of a_1 and a_2 is nonzero and there is no x such that both $b_1 = xa_1$ and $b_2 = xa_2$. Consider the quadratic expression

$$(xa_1 - b_1)^2 + (xa_2 - b_2)^2 = (a_1^2 + a_2^2)x^2 - 2(a_1 b_1 + a_2 b_2)x + (b_1^2 + b_2^2).$$

Being a sum of two squares, this expression is nonnegative for all $x \in \mathbb{R}$; moreover, since $xa_1 - b_1$ and $xa_2 - b_2$ are never equal to zero simultaneously, it is positive for all x. It follows that its discriminant is negative:

$$\big(-2(a_1 b_1 + a_2 b_2)\big)^2 - 4(a_1^2 + a_2^2)(b_1^2 + b_2^2) < 0,$$

which implies that $(a_1 b_1 + a_2 b_2)^2 < (a_1^2 + a_2^2)(b_1^2 + b_2^2)$. ∎

As a corollary, we can obtain *the triangle inequality in the plane*:

Theorem 1.6.3. *For any $a_1, a_2, b_1, b_2 \in \mathbb{R}$,*

$$\sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2} \leq \sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2}.$$

Proof. The square of the left-hand part of the inequality is $(a_1 + b_1)^2 + (a_2 + b_2)^2 = a_1^2 + a_2^2 + 2a_1 b_1 + b_1^2 + b_2^2 + 2a_2 b_2$ and of the right-hand part is $a_1^2 + a_2^2 + b_1^2 + b_2^2 + 2\sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2)}$; since $a_1 b_1 + a_2 b_2 \leq \sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2)}$ by the Cauchy-Schwarz inequality, the left-hand part is $\leq$ than the right-hand part. ∎

The general version of the triangle inequality in the plane is: For any $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$,

$$\sqrt{(a_1 - c_1)^2 + (a_2 - c_2)^2} \leq \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} + \sqrt{(b_1 - c_1)^2 + (b_2 - c_2)^2}.$$

(For three point $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$ in the plane, the distance between A and C doesn't exceed the sum of the distances between A and B and between B and C.) It is obtained from Theorem 1.6.3 by replacing a_1 by $a_1 - b_1$, a_2 by $a_2 - b_2$, b_1 by $b_1 - c_1$, and b_2 by $b_2 - c_2$.

1.7. Natural numbers and the principle of induction

Natural (positive integer) numbers are defined in the following way:

Definition. The set $\mathbb{N}$ of natural numbers is the set of real numbers satisfying the following conditions:

(i) $1 \in \mathbb{N}$;

(ii) for any natural number $n \in \mathbb{N}$ we have $n + 1 \in \mathbb{N}$ (in short, $\mathbb{N} + 1 \subseteq \mathbb{N}$ where $\mathbb{N} + 1 = \{n + 1 : n \in \mathbb{N}\}$);

(iii) $\mathbb{N}$ is the minimal set satisfying (i) and (ii) in the following sense: if $S \subseteq \mathbb{N}$ is such that $1 \in S$ and $S + 1 \subseteq S$, then $S = \mathbb{N}$.

Does $\mathbb{N}$ exist? Yes. Indeed, let $\mathcal{M}$ be the set of all subsets $M \subseteq \mathbb{R}$ satisfying (i) and (ii), $\mathcal{M} = \{M \subseteq \mathbb{R} \mid 1 \in M \wedge M + 1 \subseteq M\}$. Then $\mathcal{M}$ is nonempty since $\mathbb{R} \in \mathcal{M}$. Let $\mathbb{N}$ be the intersection of all sets from $\mathcal{M}$, $\mathbb{N} = \bigcap \mathcal{M}$. Then $1 \in \mathbb{N}$ (since $1 \in M$ for all $M \in \mathcal{M}$), and $\mathbb{N} + 1 \subseteq \mathbb{N}$ (since $\mathbb{N} + 1 \subseteq M + 1 \subseteq M$ for all $M \in \mathcal{M}$). Now if $S \subseteq N$ is such that $1 \in S$ and $S + 1 \subseteq S$, then $S \in \mathcal{M}$, so $\mathbb{N} \subseteq S$ by the construction of $\mathbb{N}$, so $S = \mathbb{N}$.

The definition of $\mathbb{N}$ implies the following *induction principle*:

Theorem 1.7.1. *Let $P(n)$ be a statement that depends on $n \in \mathbb{N}$ such that $P(1)$ is true and whenever $P(n)$ is true $P(n+1)$ is also true. (That is, for any $n \in \mathbb{N}$, $P(n) \Rightarrow P(n+1)$.) Then $P(n)$ is true for all n.*

This principle justifies the infinite process that starts like this: "Ok, $P(1)$ is true. Since $P(1)$ is true, then $P(2)$ is also true. Since $P(2)$ is true, then $P(3)$ is true..."

Proof. Define $S = \{n \in \mathbb{N} \mid P(n) \text{ is true}\}$. Then $S \subseteq \mathbb{N}$, $1 \in S$, and if $n \in S$, then $P(n)$ is true, so $P(n+1)$ is true, so $n + 1 \in S$. Hence, by (iii), $S = \mathbb{N}$, that is, $P(n)$ is true for all $n \in \mathbb{N}$. ■

Proof by induction is the main tool for proving a statement $P(n)$ for all natural numbers n. It is applied the following way: you prove $P(1)$ (this is called *the base of induction*), then you assume that n is a natural number for which $P(n)$ is true, and, under this assumption, prove $P(n+1)$ (this is called *the inductive step*). If you succeed, you say, "By induction, $P(n)$ is true for all $n \in \mathbb{N}$".

I start with a very simple theorem, which, however, cannot be proved without induction.

Theorem 1.7.2. *All natural numbers are ≥ 1.*

Proof. For every $n \in \mathbb{N}$ define $P(n)$ to be the statement "$n \geq 1$". Then $P(1)$ is "$1 \geq 1$" and is true. (This is the base of our induction.) Let $n \in \mathbb{N}$, and assume that $P(n)$ is true, that is, $n \geq 1$. Then $n + 1 > n \geq 1$, so $n + 1 \geq 1$, so $P(n+1)$ is true. (This is the step of our induction.) Hence, by induction, $P(n)$ is true for all $n \in \mathbb{N}$, that is, $n \geq 1$ for all $n \in \mathbb{N}$. ■

We understand that the natural numbers "increase": $1 < 2 < 3 < \cdots$, and in general, $n < n+1$ for all n. But do they increase unboundently, or there are real numbers so large that cannot be reached by natural numbers? *The Archimedian property* of natural numbers says that the former is true:

Theorem 1.7.3. $\mathbb{N}$ *is unbounded above.*

Proof. Let's assume, by the way of contradiction, that $\mathbb{N}$ is bounded above: there is $a \in \mathbb{R}$ such that $n \leq a$ for all $n \in \mathbb{N}$. Then, by the axiom of completeness, $\mathbb{N}$ has supremum; let $b = \sup \mathbb{N}$. Since $b - 1 < b$, $b - 1$ is not an upper bound of $\mathbb{N}$, thus there is $n \in \mathbb{N}$ such that $n > b - 1$. But then $n + 1 > b$, and $n + 1 \in \mathbb{N}$, contradiction. ■

The following example shows that we were not be able prove the Archimedian property without the axiom of completeness. Let F be the field of rational functions,

$$F = \left\{\tfrac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0} : a_i, b_j \in \mathbb{R},\ a_n \neq 0,\ b_m > 0\right\} \cup \{0\}.$$

Introduce an order on F by defining the set P of "positive elements" of F by

$$P = \left\{\tfrac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0} : a_i, b_j \in \mathbb{R},\ a_n, b_m > 0\right\}.$$

Then F is an ordered field (the axioms (P1)-(P12) can be checked) in which $\mathbb{N}$ is bounded above by, say x: we have $x - n \in P$ for all $n \in \mathbb{N}$.

As a corollary, we can obtain that numbers of the form $1/n$ with $n \in \mathbb{N}$ are "arbitrarily small" in the following sense:

Theorem 1.7.4. *For any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $0 < 1/n < \varepsilon$.*

(No real number is "arbitrarily small" by itself, a number is just a number; we say that elements of a set S of positive real numbers are *arbitrarily small* if for any $\varepsilon > 0$ there is $s \in S$ such that $0 < s < \varepsilon$.)

Proof. Given $\varepsilon > 0$, find $n \in \mathbb{N}$ such that $n > 1/\varepsilon$, then $0 < 1/n < \varepsilon$. ■

Next, let's prove, by induction, that the sum, the product, and the difference (if positive) of two natural numbers are natural numbers as well:

Theorem 1.7.5. *For any $n, m \in \mathbb{N}$ one has $n + m, nm \in \mathbb{N}$. If, in addition, $m < n$, then $n - m \in \mathbb{N}$.*

Proof. Let $m \in \mathbb{N}$ (in other words, fix m) and use induction on n. For $n = 1$ we have $n + m = m + 1 \in \mathbb{N}$. If $n \in \mathbb{N}$ satisfies $n + m \in \mathbb{N}$, then $(n + 1) + m = (n + m) + 1 \in \mathbb{N}$. By induction, $n + m \in \mathbb{N}$ for all n.

For $n = 1$ we have $nm = m \in \mathbb{N}$. If $n \in \mathbb{N}$ satisfies $nm \in \mathbb{N}$, then $(n + 1)m = nm + m \in \mathbb{N}$ as proven above. By induction, $nm \in \mathbb{N}$ for all n.

For the last statement, if $n = 1$ then the statement is true since there is no $m \in \mathbb{N}$ such that $m < n$. Assume that the statement is true for some n, that is, for any $m \in \mathbb{N}$ with $m < n$ we have $n - m \in \mathbb{N}$. Now let $m \in \mathbb{N}$, $m < n + 1$. If $m = 1$, then $(n + 1) - m = n \in \mathbb{N}$. If $m \geq 2$, then $m = k + 1$ for some $k \in \mathbb{N}$. (This also should be proved by induction.) Then $k < n$, so $n - k \in \mathbb{N}$, and so $(n + 1) - m = n - k \in \mathbb{N}$. By induction,the statement is true for all n. ■

In particular, this implies that if $n, m \in \mathbb{N}$ and $n > m$, then $n \geq m + 1$.

Induction can be used not only for proving, but also for "defining": if we define some "object" $C(1)$ and define $C(n + 1)$ assuming that $C(n)$ is already defined, then we get $C(n)$ defined for all $n \in \mathbb{N}$. This is called *an inductive*, or *recursive* definition. This is how we introduce integer *powers* of a real number: for $a \in \mathbb{R}$ we define $a^1 = a$ and $a^{n+1} = a^n a$ for all n. (So, $a^2 = a^1 a = aa$, $a^3 = a^2 a = aaa$, ...)

Theorem 1.7.6. (i) *For any $a \in \mathbb{R}$ and $n, m \in \mathbb{N}$ we have $a^{n+m} = a^n a^m$.*
(ii) *For any $a \in \mathbb{R}$ and $n, m \in \mathbb{N}$ we have $(a^n)^m = a^{nm}$.*
(iii) *For any $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$ we have $(ab)^n = a^n b^n$.*
(iv) *For any $a \in \mathbb{R}$ and $n \in \mathbb{N}$ we have $(1/a)^n = 1/a^n$.*

Proof. Let's fix m and use induction on n. For $n = 1$, $a^{n+m} = a^{m+1} = a^m a = a^m a^n$. Assume that for some n, $a^{n+m} = a^n a^m$; then $a^{(n+1)+m} = a^{n+m} a = a^n a^m a = a^{n+1} a^m$. So by induction, $a^{n+m} = a^n a^m$ for all $n, m \in \mathbb{N}$.

I leave the proof of (ii)-(iv) to you. ■

Using induction, we can obtain *Bernoulli's inequality*:

Theorem 1.7.7. *For any $x > -1$ and any $n \in \mathbb{N}$, $(1 + x)^n \geq 1 + nx$, with equality holding iff $x = 0$ or $n = 1$.*

Proof. Let $x > -1$, then $1 + x > 0$. For $n = 1$ we have an equality, $(1 + x)^1 = 1 + 1x$. Assume that for some $n \in \mathbb{N}$, $(1 + x)^n \geq 1 + nx$. Then

$$(1 + x)^{n+1} = (1 + x)^n (1 + x) \geq (1 + nx)(1 + x) = 1 + nx + x + nx^2 = 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x,$$

with equality iff $x = 0$. So, by induction, the inequality holds for all n. ■

As a corollary, we get some information about "the behavior" of powers of real numbers:

Theorem 1.7.8. *If $a > 1$, then for any $b \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $a^n > b$. If $0 < a < 1$, then for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $0 < a^n < \varepsilon$.*

Proof. Let $a > 1$. Put $x = a - 1$, then $x > 0$. For any $n \in \mathbb{N}$ we have $a^n = (1 + x)^n \geq 1 + nx$. Now, given any $b \in \mathbb{R}$, by the Arhimedian property there is $n \in \mathbb{N}$ such that $n > (b - 1)/x$, and for this n we have $1 + nx > b$, and so $a^n > b$.

Now let $0 < a < 1$, and let $\varepsilon > 0$. Then $a^{-1} > 1$, so there is $n \in \mathbb{N}$ such that $(a^{-1})^n > 1/\varepsilon$, and since $(a^n)^{-1} = (a^{-1})^n$, we obtain that $a^n < \varepsilon$. ■

Finite sums and finite products are also defined inductively. If $a_1, a_2, \ldots$ are real numbers, then we define $\sum_{i=1}^{1} a_i = a_1$, and for any n, $\sum_{i=1}^{n+1} a_i = \sum_{i=1}^{n} a_i + a_{n+1}$; we can also write $\sum_{i=1}^{n} a_i$ as $a_1 + a_2 + \cdots + a_n$. Similarly, we define $\prod_{i=1}^{1} a_i = a_1$, and for any n, $\prod_{i=1}^{n+1} a_i = \left(\prod_{i=1}^{n} a_i\right) a_{n+1}$; we can also write $\prod_{i=1}^{n} a_i$ as $a_1 a_2 \cdots a_n$.

Examples. (i) *n factorial*, $n!$, is defined as $\prod_{i=1}^{n} i = 1 \cdot 2 \cdots n$.

(ii) For any $n \in \mathbb{N}$, $\sum_{i=1}^{n} i = 1+2+\cdots+n = \frac{n(n+1)}{2}$. Indeed, this is true for $n=1$, since $1 = \frac{1(1+1)}{2}$, and if this is true for some n, then it is true for $n+1$:

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1) = \tfrac{n(n+1)}{2} + n + 1 = \tfrac{n(n+1)+2(n+1)}{2} = \tfrac{(n+1)(n+2)}{2}.$$

(iii) For any $r \in \mathbb{R} \setminus \{1\}$ and any $n \in \mathbb{N}$, $1 + a + \cdots + a^n = \frac{a^{n+1}-1}{a-1}$. As the base of induction I'll take $n = 0$, for which $1 = \frac{a-1}{a-1}$ is correct. Assume that the equality holds for some n. Then

$$1 + a + \cdots + a^n + a^{n+1} = \frac{a^{n+1}-1}{a-1} + a^{n+1} = \frac{a^{n+1} - 1 + a^{n+2} - a^{n+1}}{a-1} = \frac{a^{n+2}-1}{a-1},$$

which establishes the induction step.

The following properties of finite sums are easy to prove by induction on n:

Theorem 1.7.9. *For any $n \in \mathbb{N}$ and $a_1, \ldots, a_n, b_1, \ldots, b_n, c \in \mathbb{R}$, $\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$, $\sum_{i=1}^{n}(ca_i) = c\sum_{i=1}^{n} a_i$, and $\sum_{i=1}^{n}(a_i+b_i)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n} a_ib_i + \sum_{i=1}^{n} b_i^2$.*

Actually, an induction process may start not at 1 but at any other natural (in fact, any integer) number:

Theorem 1.7.10. *Let $P(n)$ be a statement that depends on $n \in \mathbb{N}$ such that $P(n_0)$ is true for some $n_0 \in \mathbb{N}$ and for any n, $P(n) \Rightarrow P(n+1)$. Then $P(n)$ is true for all integer $n \geq n_0$.*

To prove this, we apply the conventional induction to the statement $P'(n) =$ "$n < n_0$ or $P(n)$ is true".

Example. We may use this "modified" induction principle to prove that $n! > 2^n$ for all $n \geq 4$. Indeed, for $n = 4$ we have $4! = 24 > 16 = 2^4$, and if $n! > 2^n$ for some $n \in \mathbb{N}$, then $(n+1)! = n!(n+1) > 2^n 2 = 2^{n+1}$.

Binomial coefficients are also defined inductively: for any $n, k \in \mathbb{N} \cup \{0\}$ with $k \leq n$ we define $\binom{n}{k}$ (read as "n choose k") in the following way: for any n, if $k = 0$ or $k = n$ then $\binom{n}{k} = 1$, and for any n and $1 \leq k \leq n$, $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. We may visualize this definition using the so-called *Pascal triangle*:

$$\begin{array}{ccccccccccc}
&&&&& 1 &&&&& \\
&&&& 1 && 1 &&&& \\
&&& 1 && 2 && 1 &&& \\
&& 1 && 3 && 3 && 1 && \\
& 1 && 4 && 6 && 4 && 1 & \\
1 && 5 && 10 && 10 && 5 && 1 \\
\vdots\vdots && \vdots && \vdots && \vdots && \vdots && \vdots\vdots
\end{array}
\qquad
\begin{array}{ccccccccccc}
&&&&& \binom{0}{0} &&&&& \\
&&&& \binom{1}{0} && \binom{1}{1} &&&& \\
&&& \binom{2}{0} && \binom{2}{1} && \binom{2}{2} &&& \\
&& \binom{3}{0} && \binom{3}{1} && \binom{3}{2} && \binom{3}{3} && \\
& \binom{4}{0} && \binom{4}{1} && \binom{4}{2} && \binom{4}{3} && \binom{4}{4} & \\
\binom{5}{0} && \binom{5}{1} && \binom{5}{2} && \binom{5}{3} && \binom{5}{4} && \binom{5}{5} \\
\vdots\vdots && \vdots && \vdots && \vdots && \vdots && \vdots\vdots
\end{array}$$

where each entry is the sum of the two entries right above it. The rows of the table and the entries in every row are enumerated starting from 0; the k-th entry in the n-th row is just $\binom{n}{k}$. (One can prove that for any n amd k, $\binom{n}{k}$ is the number of k-element subsets in an n-element set.)

We define $0! = 1$. We can now obtain a (non-inductive) formula for binomial coefficients:

Theorem 1.7.11. *For all $n, k \in \mathbb{N} \cup \{0\}$ with $k \leq n$, $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$.*

Proof. Of course, we use induction. For $n = 1$, $\binom{1}{0} = 1 = \frac{1!}{0!1!}$ and $\binom{1}{1} = 1 = \frac{1!}{1!0!}$ are true. Assume that for some $n \in \mathbb{N}$, for all integer k with $0 \leq k \leq n$ we have $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Then for all integer k with $1 \leq k \leq n$ we have

$$\begin{aligned}\tbinom{n+1}{k} = \tbinom{n}{k} + \tbinom{n}{k-1} = \tfrac{n!}{k!(n-k)!} + \tfrac{n!}{(k-1)!(n-k+1)!} &= \tfrac{n!}{k(k-1)!(n-k)!} + \tfrac{n!}{(k-1)!(n-k)!(n-k+1)} \\ &= n!\tfrac{(n-k+1)+k}{k(k-1)!(n-k)!(n-k+1)} = n!\tfrac{n+1}{k!(n-k+1)!} = \tfrac{(n+1)!}{k!(n-k+1)!}.\end{aligned}$$

For $k = 0$ and $k = n+1$ we also have $\binom{n+1}{0} = 1 = \frac{(n+1)!}{0!(n+1)!}$ and $\binom{n+1}{n+1} = 1 = \frac{(n+1)!}{(n+1)!0!}$. So, by induction, the formula holds for all $n \in \mathbb{N}$ and all integer k with $0 \leq k \leq n$. ∎

Newton's binomial formula is the following theorem:

Theorem 1.7.12. *For any $a, b \in \mathbb{R}$ and any $n \in \mathbb{N}$,*

$$(a+b)^n = \sum_{k=0}^{n} \tbinom{n}{k} a^k b^{n-k}.$$

Proof. Let $a, b \in \mathbb{R}$. For $n = 1$ the identity holds:

$$(a+b)^1 = a+b = 1a+1b = \tbinom{1}{0}a^0b^1 + \tbinom{1}{1}a^1b^0 = \sum_{k=0}^{1} \tbinom{1}{k} a^k b^{n-k}.$$

Assume that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ for some $n \in \mathbb{N}$. Then

$$\begin{aligned}(a+b)^{n+1} = (a+b)^n(a+b) = \Big(\sum_{k=0}^{n} \tbinom{n}{k} a^k b^{n-k}\Big)(a+b) &= \sum_{k=0}^{n} \tbinom{n}{k}\big(a^{k+1}b^{n-k} + a^k b^{n-k+1}\big) \\ &= \sum_{k=0}^{n} \tbinom{n}{k} a^{k+1}b^{n-k} + \sum_{k=0}^{n} \tbinom{n}{k} a^k b^{n-k+1}\end{aligned}$$

In the first sum we may replace k by $l-1$ (then $l = k+1$ will range between 1 and $n+1$): $\sum_{k=0}^{n} \binom{n}{k} a^{k+1}b^{n-k} = \sum_{l=1}^{n+1} \binom{n}{l-1} a^l b^{n-l+1}$, and then simply replace l by k, which affects nothing: $\sum_{k=1}^{n+1} \binom{n}{k-1} a^k b^{n-k+1}$. So, we get:

$$\begin{aligned}(a+b)^{n+1} &= \sum_{k=1}^{n+1} \tbinom{n}{k-1} a^k b^{n-k+1} + \sum_{k=0}^{n} \tbinom{n}{k} a^k b^{n-k+1} \\ &= \sum_{k=1}^{n} \tbinom{n}{k-1} a^k b^{n-k+1} + \tbinom{n}{n} a^{n+1}b^0 + \tbinom{n}{0} a^0 b^{n+1} + \sum_{k=1}^{n} \tbinom{n}{k} a^k b^{n-k+1} \\ &= a^0b^{n+1} + \sum_{k=1}^{n} \big(\tbinom{n}{k-1} + \tbinom{n}{k}\big) a^k b^{n-k+1} + a^{n+1}b^0 \\ &= \tbinom{n+1}{0} a^0 b^{n+1} + \sum_{k=1}^{n} \tbinom{n+1}{k} a^k b^{n-k+1} + \tbinom{n+1}{n+1} a^{n+1} b^0 = \sum_{k=0}^{n+1} \tbinom{n+1}{k} a^k b^{n+1-k},\end{aligned}$$

that is, the identity also holds for $n+1$. So, by induction, it holds for all $n \in \mathbb{N}$. ■

In an induction process, proving that a statement is true for $n+1$, we assume that it is true for n; or equivalently, proving that it is true for n, we assume that it is true for $n-1$. Actually, we may as well assume that it is true not only for $n-1$, but for all natural numbers $k < n$; this is called *the principle of complete*, or *strong induction*:

Theorem 1.7.13. *Let $P(n)$, $n \in \mathbb{N}$, be a statement that depends on $n \in \mathbb{N}$ with the property that for any $n \in \mathbb{N}$, if $P(k)$ is true for all $k \in \mathbb{N}$ such that $k < n$, then $P(n)$ is also true. (That is, assuming that $P(k)$ is true for all $k \in \mathbb{N}$ with $k < n$, you can prove $P(n)$.) Then $P(n)$ is true for all $n \in \mathbb{N}$.*

Proof. We will deduce this principle from the "ordinary" principle of induction. For every $n \in \mathbb{N}$ let $Q(n)$ be the statement "$P(k)$ is true for all natural $k \leq n$". For $n = 1$, since there is no natural $k < 1$, $P(1)$ is true without any assumptions, and so $Q(1)$ is true. Assume that $Q(n)$ is true for some n. Then $P(k)$ is true for all natural $k \leq n$, that is, for all natural $k < n+1$. Hence, by our assumption, $P(n+1)$ is true. So, $P(k)$ is true for all natural $k \leq n+1$. So, $Q(n+1)$ is true. By (ordinary) induction, $Q(n)$ is true for all $n \in \mathbb{N}$. Since $Q(n)$ implies $P(n)$, $P(n)$ is also true for all $n \in \mathbb{N}$. ■

We say that an ordered set X is *well ordered* if every nonempty subset of X has the least element. As an application of the principle of complete induction, we can prove that $\mathbb{N}$ is well ordered:

Theorem 1.7.14. *Every nonempty subset of $\mathbb{N}$ has the least element.*

Proof. I'll apply a *proof by contraposition*: instead of proving "if S is nonempty then S has the least element", I'll prove *the contrapositive*: "If S doesn't have the least element then S is empty". I'll use the principle of complete induction: Let $n \in \mathbb{N}$ and assume that for all natural $k < n$ we have $k \notin S$. Then if $n \in S$, then n is the least element of S; by our assumption this is not the case, so $n \notin S$. Hence, by complete induction, $n \notin S$ for all $n \in \mathbb{N}$, that is, $S = \emptyset$. ■

1.8. Integers, divisibility and primes

A number $n \in \mathbb{R}$ is said to be *integer*, or *an integer*, if $n \in \mathbb{N}$, or $n = 0$, or $-n \in \mathbb{N}$. The set of integers is denoted by $\mathbb{Z}$, $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup (-\mathbb{N})$, where $-\mathbb{N} = \{-n \mid n \in \mathbb{N}\} = \{n \mid -n \in \mathbb{N}\}$.

Theorem 1.8.1. *$\mathbb{Z}$ is closed under addition, subtraction, and multiplication: for any $n, m \in \mathbb{Z}$ we have $n + m, n - m, nm \in \mathbb{Z}$.*

Proof. Let $n, m \in \mathbb{Z}$.

Since $\mathbb{Z}$ is defined "by cases", to prove that $n + m \in \mathbb{Z}$ we consider a few cases:
if one of n, m is 0, w.l.o.g. $m = 0$, then $n + m = n \in \mathbb{Z}$;
if both n, m are positive, $n, m \in \mathbb{N}$, then $n + m \in \mathbb{N}$ was proven, so $n + m \in \mathbb{Z}$;
if both n, m are negative, $-n, -m \in \mathbb{N}$, then $n + m = -((-n) + (-m))$ and $(-n) + (-m) \in \mathbb{N}$ so $n + m \in \mathbb{Z}$;
if one of n, m is positive and the other is negative, we may assume w.l.o.g. that $n \in \mathbb{N}$ and $-m \in \mathbb{N}$; then if $n > -m$ then $n + m = n - (-m) \in \mathbb{N}$ was proven, and if $-m > n$ then $n + m = -((-m) - n)$ and $(-m) - n \in \mathbb{N}$, so $n + m \in \mathbb{Z}$.

If $m \in \mathbb{Z}$ then $-m \in \mathbb{Z}$, so $n - m = n + (-m) \in \mathbb{Z}$.

If $n, m \in \mathbb{N}$ then $nm \in \mathbb{N}$;
if $n, -m \in \mathbb{N}$ then $nm = -(n(-m))$ and $n(-m) \in \mathbb{N}$, so $nm \in \mathbb{Z}$;
the case $-n, m \in \mathbb{N}$ is similar;
if $-n, -m \in \mathbb{N}$, then $nm = (-n)(-m) \in \mathbb{N}$. ■

Theorem 1.8.1 says that $\mathbb{Z}$ is *a ring*. Moreover, $\mathbb{Z}$ is *an ordered ring*, meaning that it satisfies all axioms (P1)-(P13) except (P8). (The elements of $\mathbb{Z}$, except ± 1, have no multiplicative inverses in $\mathbb{Z}$.)

$\mathbb{Z}$ is not well ordered, $\mathbb{Z}$ itself has no minimal element. However, it has the following property:

Theorem 1.8.2. *Every nonempty bounded below subset of $\mathbb{Z}$ has a least element; every nonempty bounded above subset of $\mathbb{Z}$ has a greatest element.*

Proof. Let S be a nonempty bounded below subset of $\mathbb{Z}$. Let $b \in \mathbb{R}$ be such that $b \leq n$ for all $n \in S$. Let $m \in \mathbb{N}$ be such that $m > -b$, then $-m < b \leq n$ for all $n \in S$. Then $n + m \in \mathbb{Z}$ and $n + m > 0$ for all $n \in S$, so $S + m = \{n + m \mid n \in S\} \subseteq N$. Let n_0 be the minimal element of $S + m$, then $n_0 - m$ is the minimal element of S.

Let S be a nonempty bounded above subset of $\mathbb{Z}$. Then $-S = \{-n \mid n \in S\}$ is a nonempty bounded below subset of $\mathbb{Z}$. Let n_0 be the least element of $-S$; then $-n_0$ is the greatest element of S. ■

For every $a \in \mathbb{R}$ there exists $n \in \mathbb{Z}$ such that $n \leq a < n + 1$. Indeed, let $S = \{n \in \mathbb{Z} : n \leq a\}$; then S is a nonempty bounded above (by a) subset of $\mathbb{Z}$, so it has a maximal element n. We then have $n \leq a$ and $n + 1 > a$. This n is called *the integer part* of a and is denoted by $[a]$, the number $a - [a] \in [0, 1)$ is called *the fractional part* of a and is denoted by $\{a\}$.

For two integers n and d we say that d *divides* n, or n *is divisible by* d, and write $d \mid n$ if there is $k \in \mathbb{Z}$ such that $n = dk$. (In other words, if $n/d \in \mathbb{Z}$, or $n = d = 0$.) By this definition, 0 is divisible by all integers, and 1, -1 divide all integers. If $m \mid n$, then $|m| \leq |n|$; it follows that 1 is only divisible by 1 and -1.

Theorem 1.8.3. *For any $d, n, m \in \mathbb{Z}$, if $d \mid n$ then $-d \mid n$, $d \mid -n$, $-d \mid -n$; if $d \mid n$ and $d \mid m$ then $d \mid n + m$ and $d \mid n - m$; if $d \mid n$ and $n \mid m$ then $d \mid m$; if $d \mid n$ then $d \mid rn$ for all $r \in \mathbb{Z}$.*

Proof. All this follows directly from the definition: if $n = kd$ then $-n = (-k)d$, $n = (-k)(-d)$, $-n = k(-d)$; if $n = kd$ and $m = ld$, then $n + m = (k + l)d$, $b - m = (k - l)d$; if $n = kd$ and $m = ln$, then $m = (kl)d$; if $n = kd$ then $rn = (rk)d$. ■

We say that an integer n is *even* if $2 \mid n$ and *odd* otherwise. Even and odd integers are described by the following theorem:

Theorem 1.8.4. *Every integer n has form $n = 2k$ (and is even in this case) or $n = 2k + 1$ (and is odd in this case) for some $k \in \mathbb{Z}$.*

Proof. If $n = 2k$ for some $k \in \mathbb{Z}$, then n is even by definition; if $n = 2k + 1$ for some $k \in \mathbb{Z}$, it is odd since $2 \mid 2k$ and $2 \nmid 1$, so $2 \nmid (2k + 1)$.

$1 = 2 \cdot 0 + 1$; if $n = 2k$ then $n + 1 = 2k + 1$; if $n = 2k + 1$, then $n + 1 = 2(k + 1)$. By induction, this proves the assertion for all $n \in \mathbb{N}$. Now, $0 = 2 \cdot 0$, and for $n \in \mathbb{Z}$ with $n < 0$, if $-n = 2k$ then $n = 2(-k)$, and if $-n = 2k + 1$, then $n = -2k - 1 = 2(-k - 1) + 1$.

Two integers n and m are said to be *coprime* if they have no common divisors except 1 and -1: if $d \mid n$ and $d \mid m$, then $d = 1$ or $d = -1$.

Theorem 1.8.5. *$n, m \in \mathbb{Z}$ are coprime iff there are $k, l \in \mathbb{Z}$ such that $kn + lm = 1$.*

Proof. Assume that there are k, l such that $kn + lm = 1$. If $d \mid n, m$, then $d \mid 1$, so $d = \pm 1$; hence, n, m are coprime in this case.

Assume that n, m are coprime. W.l.o.g. we may assume that $n, m \in \mathbb{N}$. (Indeed, 0 is not coprime with any integer, and if, say, $n < 0$, we may replace it by $-n$ (and k by $-k$) and nothing changes.) I'll use complete induction on $\max\{n, m\}$. If $n = m$, then n, m can only be coprime if $n, m = \pm 1$, and then $1n + 0m = 1$ or $(-1)n + 0m = 1$. W.l.o.g. assume that $n > m$. Then $n - m$ and m are coprime. (If $n - m$ and m have a common divisor d, then d also divides n.) By complete induction principle we may assume that there are $k, l \in \mathbb{N}$ such that $k(n - m) + lm = 1$. Then $kn + (l - k)m = 1$, and $k, l - k \in \mathbb{Z}$. ■

A positive integer $p \in \mathbb{N} \setminus \{1\}$ is said to be *prime* or *a prime* if p has no divisors except ± 1 and $\pm p$. (That is, if $d \mid p$, then $d = 1$ or $d = -1$ or $d = p$ or $d = -p$.) It is clear that if p is prime, then for any $n \in \mathbb{Z}$, n and p are not coprime iff $p \mid n$.

Prime numbers have the following nice property (which, actually, characterizes them):

Theorem 1.8.6. *If p is prime, $n, m \in \mathbb{Z}$, and $p \mid nm$, then $p \mid n$ or $p \mid m$.*

Proof. Assume that $p \mid nm$ and $p \nmid n$. Then p, n are coprime; let $k, l \in \mathbb{Z}$ be such that $kn + lp = 1$. Then $knm + lpm = m$. Since $p \mid nm$, we have $p \mid knm$, and clearly $p \mid lpm$, so $p \mid (knm + lpm)$, so $p \mid m$. ■

Using induction, we may generalize this theorem:

Theorem 1.8.7. *If p is prime, $n_1, \ldots, n_k \in \mathbb{N}$, and $p \mid (n_1 \cdots n_k)$, then there is i with $1 \le i \le k$ such that $p \mid n_i$.*

Proof. I'll use induction on k, the number of multipliers. This is clearly true if $k = 1$ (and is already proved for $k = 2$). Assume it is true for some k, and let $p \mid (n_1 \cdots n_k n_{k+1})$. By Theorem 1.8.6, $p \mid (n_1 \cdots n_k)$ or $p \mid n_{k+1}$. If $p \mid n_{k+1}$ we are done; if $p \mid (n_1 \cdots n_k)$, then $p \mid n_i$ for some i by our induction hypothesis. ■

We are now in position to prove *the Fundamental Theorem of Arithmetic*:

Theorem 1.8.8. *Every integer $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$ is (up to the sign) representable as a product, $n = \pm p_1 \cdots p_k$, of primes. This representation is unique up to permutation of factors: if $n = \pm p_1 \cdots p_k = \pm q_1 \cdots q_l$ where $p_1, \ldots, p_k, q_1, \ldots, q_l$ are prime, then $k = l$ and after a reordering, if necessary, of $q_1, \ldots, q_l$, $p_i = q_i$ for all i.*

Proof. W.l.o.g., we may assume that $n \in \mathbb{N}$.
Existence: I'll use complete induction on n. If n is prime, we are done. If n is not prime, $n = md$ with $m, d \in \mathbb{N}$, $m, d < n$. By complete induction assumption, the statement holds for m and d: $m = p_1 \cdots p_k$ and $d = q_1 \cdots q_l$ where $p_1, \ldots, p_k, q_1, \ldots, q_l$ are prime. Then $n = p_1 \cdots p_k q_1 \cdots q_l$, and we are done.
Uniqueness: Let $n = \pm p_1 \cdots p_k = \pm q_1 \cdots q_l$ where $p_1, \ldots, p_k, q_1, \ldots, q_l$ are prime. Then $p_1 \mid q_1 \cdots q_l$, so $p_1 \mid q_j$ for some j, but q_j is prime, so $p_1 = q_j$. Let's renumerate $q_1, \ldots, q_l$ so that $j = 1$, so that $p_1 = q_1$. Then $p_1 p_2 \cdots p_k = p_1 q_2 \cdots q_l$. I'll use induction on k; if $k = 1$ this means that $p_1 = p_1 q_2 \cdots q_l$, which implies that $l = 1$ (and $p_1 = q_1$). If $k \ge 2$, then we have $p_2 \cdots p_k = q_2 \cdots q_l$; by induction on k we may assume that $k - 1 = l - 1$, so $k = l$, and that after a renumeration, $p_2 = q_2$, …, $p_k = q_k$. ■

1.9. Intervals, neighborhoods, infinite points, and dense sets

For $a, b \in \mathbb{R}$ with $a < b$ we define $[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$, $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$, $[a,b) = \{x \in \mathbb{R} \mid a \le x < b\}$, and $(a,b] = \{x \in \mathbb{R} \mid a < x \le b\}$; these sets are called *intervals* with *endpoints* a and b, $[a,b]$ is called *a closed interval* and (a,b) is *an open interval*. *The length* $|I|$ of such an interval I is defined to be $b - a$.

It is an easy but important fact: for any interval I and any $x, y \in I$ we have $|y - x| \le |I|$, and if I is open, then, moreover, $|y - x| < |I|$. Indeed, if $a \le x \le y \le b$, then $|y - x| = y - x \le b - a = |I|$; if $a \le y \le x \le b$, then $|y - x| = x - y \le b - a = |I|$, etc.

The following simple lemma is also worth mentioning:

Lemma 1.9.1. *If $[a,b]$ is an interval in $\mathbb{R}$, $n \in \mathbb{N} \setminus \{1\}$ and S is an n-element subset of $[a,b]$, then there are distinct elements $x, y \in S$ such that $|x - y| \le (b-a)/(n-1)$. If S is an infinite subset of $[a,b]$ then for any $\varepsilon > 0$ there are distinct $x, y \in S$ such that $|x - y| < \varepsilon$.*

Proof. Let $S = \{x_1, \dots, x_n\}$, and let $x_1 < \dots < x_n$. If $x_{i+1} - x_i > (b-a)/(n-1)$ for all $i = 1, \dots, n-1$, then $x_n - x_1 = \sum_{i=1}^{n-1}(x_{i+1} - x_i) > (n-1)(b-a)/(n-1) = b - a$, contradiction.

If S is infinite and $\varepsilon > 0$ is given, find $n \in \mathbb{N}$ such that $(b-a)/(n-1) < \varepsilon$, choose an n-elements subset S' of S, and choose distinct elements x and y of S' (and so, of S) such that $|x - y| \le (b-a)/(n-1)$. ■

Given a point $a \in \mathbb{R}$ and $\varepsilon > 0$, the interval $(a - \varepsilon, a + \varepsilon)$ is called *the ε-neighborhood* of a; we have $x \in (a - \varepsilon, a + \varepsilon)$ iff $|x - a| < \varepsilon$. The interval $(a - \varepsilon, a]$ is *a left-hand neighborhood* of a, the interval $[a, a + \varepsilon)$ is *a right-hand neighborhood* of a.

We add two *infinite* points to $\mathbb{R}$, $+\infty$ and $-\infty$, and call the obtained set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ *the extended real line*. We extend the order on $\mathbb{R}$ to an order on $\overline{\mathbb{R}}$ by putting $-\infty < a < +\infty$ for all $a \in \mathbb{R}$. Thus, $-\infty$ is the minimal point of $\overline{\mathbb{R}}$ and $+\infty$ is the maximal point, and $\overline{\mathbb{R}}$ is the closed interval $[-\infty, +\infty]$. We also have two half-closed intervals $(-\infty, +\infty]$ and $[-\infty, +\infty)$ and the open interval $\mathbb{R} = (-\infty, +\infty)$. The operations of addition, multiplication and division extend to $\overline{\mathbb{R}}$ only partially (so that $\overline{\mathbb{R}}$ is not a field under these operations): we have $(\pm\infty) + a = \pm\infty$ for all $a \in \mathbb{R}$; $(+\infty) + (+\infty) = +\infty$ and $(-\infty) + (-\infty) = -\infty$; $(\pm\infty)a = \pm\infty$ for all $a > 0$ and $(\pm\infty)a = \mp\infty$ for all $a < 0$; $(+\infty) \cdot (+\infty) = (-\infty) \cdot (-\infty) = +\infty$ and $(+\infty) \cdot (-\infty) = (-\infty) \cdot (+\infty) = -\infty$; $1/(\pm\infty) = 0$. The results of the operations $(+\infty) + (-\infty)$, $(\pm\infty) \cdot 0$, $(\pm\infty)/(\pm\infty)$ are not defined.

Alternatively, we can "complete" $\mathbb{R}$ by adding only one point ∞, which represents both $+\infty$ and $-\infty$. The set $\mathbb{R} \cup \{\infty\}$ is not ordered. (It can be viewed as a circle, obtained from $\overline{\mathbb{R}}$ by glueing its ends together.) The operation of addition is not well with ∞; multiplication is partially defined by $\infty \cdot a = \infty$ for all $a \ne 0$, $\infty \cdot \infty = \infty$, $1/0 = \infty$, $1/\infty = 0$.

An interval $(a, +\infty)$ can be seen as a neighborhood of $+\infty$, an interval $(-\infty, a)$ as a neighborhood of $-\infty$. The union $(-\infty, a) \cup (b, +\infty)$ is a neighborhood of ∞.

Let A be a set of real numbers. A number a is said to be *a limit point* of A if for any $\varepsilon > 0$ there exists $x \in A \setminus \{a\}$ such that $|x - a| < \varepsilon$ (that is, any neighborhood of a contains a point of A distinct from a). A limit point of A may belong and may not belong to A. If A contains all its limit points it is said to be *closed*. The set of limit points of A is denoted by A', and the set $\overline{A} = A \cup A'$ is called *the closure* of A. We have $a \in \overline{A}$ iff for any $\varepsilon > 0$ there exists $x \in A$ such that $|x - a| < \varepsilon$ (that is, any neighborhood of a contains a point of A (which can be a itself)).

A number $a \in A$ which is not a limit point of A is called *an isolated point* of A; a is isolated iff there is $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \cap A = \{a\}$.

A set A is said to be *discrete* if every point of A is isolated.

A set $A \subseteq \mathbb{R}$ is said to be *dense* in $\mathbb{R}$ if every interval in $\mathbb{R}$ contains an element of A; this means that $\overline{A} = \mathbb{R}$. More generally, let A, B be subsets of $\mathbb{R}$; we say that A *is dense* in B if $B \subseteq \overline{A}$, that is, every neighborhood of every point of B contains a point of A: for every $b \in B$ and every $\varepsilon > 0$ there exists $a \in A$ such that $|a - b| < \varepsilon$.

A subset A of $\mathbb{R}$ is said to be *nowhere dense* if it is not dense in every subinterval I of $\mathbb{R}$: there is a subinterval J of I such that $A \cap J = \emptyset$.

1.10. Rational and irrational numbers

A real number r is said to be *rational* if there are $n.m \in \mathbb{Z}$, with $m \neq 0$, such that $r = n/m$. The set of rational numbers is denoted by $\mathbb{Q}$. It is easy to prove that $\mathbb{Q}$ is closed under addition, multiplication, subtraction, and division (and so, $\mathbb{Q}$ is a field):

Theorem 1.10.1. *For any $r, s \in \mathbb{Q}$ we have $r+s, r-s, rs \in \mathbb{Q}$, and if $s \neq 0$, then also $r/s \in \mathbb{Q}$.*

Proof. Let $r = n/m$ and $s = k/l$ where $n, m, k, l \in \mathbb{Z}$, $m, l \neq 0$. Then $r + s = (nl + km)/ml \in \mathbb{Q}$ since $nl + km, ml \in \mathbb{Z}$ and $ml \neq 0$. Also, since $-s = (-k)/l \in \mathbb{Q}$, we get that $r - s \in \mathbb{Q}$ as well. Also, $rs = (nk)/(ml) \in \mathbb{Q}$ since $nk, ml \in \mathbb{Z}$, $ml \neq 0$, and if $k \neq 0$, then also $rs = (nl)/(mk) \in \mathbb{Q}$ since $nl, mk \in \mathbb{Z}$, $mk \neq 0$. ■

The representation of a rational number in the form n/m with $n, m \in \mathbb{Z}$ is not unique, since for any $k \in \mathbb{Z}\setminus\{0\}$ we have $n/m = (kn)/(km)$. However, every rational number r has "the best" such representation. Indeed, r has a representation n/m with $n \in \mathbb{Z}$, $m \in \mathbb{N}$. (If $r = n/m$ with $m < 0$, then also $r = (-n)/(-m)$.) And among all such representations of r there is one with the minimal denominator. (In more details: let $S = \big\{d \in \mathbb{N} \bigm| \text{there exists } c \in \mathbb{Z} \text{ such that } r = c/d\big\}$. Since S is nonempty, it has the least element.) This representation is called *the representation of r in lowest terms*. If n/m is a representation of a rational number in lowest terms, then n and m are coprime: indeed, if $n = n'd$ and $m = m'd$ where $d \in \mathbb{N} \setminus \{1\}$, then $n/m = n'/m'$ with $m' < m$. (The converse is also true; prove it, if you like.)

Non-rational real numbers are said to be *irrational*. Irrational numbers do exist:

Theorem 1.10.2. *$\sqrt{2}$ is irrational.*

This theorem has many different proofs. The most popular and standard proof is the following:

Proof. By the way of contradiction, assume that $\sqrt{2}$ is rational. Let $\sqrt{2} = n/m$, $n, m \in \mathbb{N}$, be the representation in lowest terms. We have $2 = n^2/m^2$, so $n^2 = 2m^2$, so $2 \bigm| n^2$. Since 2 is prime, by Theorem 1.8.6, this implies that $2 \bigm| n$, that is, $n = 2k$ for some $k \in \mathbb{Z}$. Then $4k^2 = 2m^2$, so $m^2 = 2k^2$, so $2 \bigm| m^2$, so $2 \bigm| m$, that is, $m = 2l$ for some $l \in \mathbb{Z}$. Hence, $\sqrt{2} = n/m = (2k)/(2l) = k/l$ with $k, l \in \mathbb{N}$, $l < m$. This contradicts our assumption that n/m is the representation of $\sqrt{2}$ in lowest terms. ■

Unlike rational numbers, the set of irrational numbers is not closed under addition and multiplication: $\sqrt{2}$ is irrational, and so is $-\sqrt{2}$, but $\sqrt{2} + (-\sqrt{2}) = 0$ and $\sqrt{2}\sqrt{2} = 2$ are rational. The sum (and the product, if nonzero) of a rational and an irrational numbers is always irrational: if $a \in \mathbb{Q}$, $b \in \mathbb{R} \setminus \mathbb{Q}$, then $a + b \notin \mathbb{Q}$ since otherwise $b = (a + b) - a \in \mathbb{Q}$.

Rational numbers are everywhere in $\mathbb{R}$:

Theorem 1.10.3. *$\mathbb{Q}$ is dense in $\mathbb{R}$.*

Proof. Let $a, b \in \mathbb{R}$, $a < b$; we need to show that the interval (a, b) contains a rational number, that is, there is $r \in \mathbb{Q}$ such that $a < r < b$. Find $m \in \mathbb{N}$ such that $1/m < b - a$. The set $\{n \in \mathbb{Z} : n/m > a\} = \{n \in \mathbb{Z} : n > ma\}$ is nonempty and bounded below, so it has the least element n; we then have $n/m > a$ and $(n-1)/m \leq a$. The second inequality implies that $n/m \leq a + 1/m < a + b - a = b$, so $a < n/m < b$. ■

It is easy to prove that the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is also dense in $\mathbb{R}$.

I am now going to present another proof of the fact that $\sqrt{2}$ is irrational. First, let's do a little research. Let $\alpha \in \mathbb{R} \setminus \{0\}$, and consider the set $A = \big\{n\alpha + m \bigm| n, m \in \mathbb{Z}\big\}$. A is closed under addition and subtraction: if $a, b \in A$ then $a + b, a - b \in A$. (We say that A is *a group* under addition.) If α is rational, $\alpha = k/d$ with $k \in \mathbb{Z}$, $d \in \mathbb{N}$, then all elements of A have form $(nk + md)/d = r/d$ with $r \in \mathbb{Z}$, that is, A is contained in the set $\frac{1}{d}\mathbb{Z} = \big\{\frac{r}{d} \bigm| r \in \mathbb{Z}\big\}$. Actually, if k/d is the lowest terms representation of α, then $A = \frac{1}{d}\mathbb{Z}$: indeed, there are $n, m \in \mathbb{Z}$ such that $nk + md = 1$, so A contains $\frac{1}{d}$, and so, also contains all numbers of the form $\frac{r}{d}$ with $r \in \mathbb{Z}$. In particular, A is discrete, and the interval $(0, 1/d)$ contains no elements of A.

Now, let α be irrational; I'll show that A is dense in $\mathbb{R}$. For any $a \in A$, the fractional part $\{a\} \in A$ as well; I claim that the elements $\{n\alpha\}$ of A, with $n \in \mathbb{Z}$, are all distinct. Indeed, if $\{n\alpha\} = \{n'\alpha\}$ with distinct $n, n' \in \mathbb{Z}$, then $n\alpha = n'\alpha + k$ for some $k \in \mathbb{Z}$, then $\alpha = k/(n - n') \in \mathbb{Q}$, which is not the case. Hence, the interval $[0, 1)$ contains infinitely many elements of A. By Lemma 1.9.1 for any $\varepsilon > 0$ there are $a_1, a_2 \in A$ such that $0 < a_1 - a_2 < \varepsilon$. But $a = a_1 - a_2 \in A$, so there is $a \in A$ such that $0 < a < \varepsilon$. Now let $b \in \mathbb{R}$ and

$\varepsilon > 0$, find $a \in A$ such that $0 < a < \varepsilon$, let $n = [b/a] + 1$, so that $n \in \mathbb{Z}$ and $n - 1 \leq b/a < n$; then $na > b$ and $na - a \leq b$, so $na \leq b + a < b + \varepsilon$. Hence, $na \in (b, b + \varepsilon)$; since $na \in A$, this proves that A is dense in $\mathbb{R}$.

We've just proved:

Theorem 1.10.4. *Let $\alpha \in \mathbb{R} \setminus \{0\}$ and $A = \{n\alpha + m \mid n, m \in \mathbb{Z}\}$. If $\alpha \in \mathbb{Q}$, then $A = \frac{1}{d}\mathbb{Z}$ for some $d \in \mathbb{N}$. If α is irrational, then A is dense in $\mathbb{R}$.*

Using this result we can now reprove that $\sqrt{2}$ is irrational:

Proof of Theorem 1.10.2. Let $A = \{n\sqrt{2} + m \mid n, m \in \mathbb{Z}\}$. Notice that A is closed not only under addition, but also under multiplication (is a ring): for any $n, m, k, l \in \mathbb{Z}$, $(n\sqrt{2} + m)(k\sqrt{2} + l) = (nl + mk)\sqrt{2} + (2nk + ml) \in A$. Since $1^2 < 2 < 2^2$, we have that $1 < \sqrt{2} < 2$, so $0 < \sqrt{2} - 1 < 1$. It follows (by Theorem 1.7.8) that for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $0 < (\sqrt{2} - 1)^n < \varepsilon$. By Theorem 1.10.4, $\sqrt{2}$ is irrational. ■

Same proof shows that for any $D \in \mathbb{N}$, either $\sqrt{D} \in \mathbb{N}$ or $\sqrt{D}$ is irrational.

1.11. Supremum and infimum

Recall that for any nonempty bounded above set $A \subseteq \mathbb{R}$, the supremum of A, $\sup A$, is the lowest upper bound of A; the existence of $\sup A$ is declared by the axiom of completeness. If A is unbounded above, we write $\sup A = +\infty$.

Let a nonempty $A \subseteq \mathbb{R}$ be bounded above. If A has the greatest element b, then $\sup A = b$. Indeed, we have $a \leq b$ for all $a \in A$, so b is an upper bound of A. On the other hand, any $c < b$ is not an upper bound of A since $b \in A$ and $b > c$, so b is the least upper bound of A. If A has no greatest element, then $\sup A \notin A$.

We have that $b = \sup A$ if $b \geq a$ for all $a \in A$ and for any $c < b$, c is not an upper bound of A, that is, there is $a \in A$ such that $a > c$. If $b \in \mathbb{R}$, then we have $c < b$ iff $c = b - \varepsilon$ for some $\varepsilon > 0$ (namely, for $\varepsilon = b - c$). Thus, the definition of supremum can be rewritten in the following way:

Theorem 1.11.1. *Let $A \subseteq \mathbb{R}$ be nonempty. Then $b = \sup A$ iff $a \leq b$ for all $a \in A$ and for every $c < b$ there exists $a \in A$ such that $a > c$. If A is bounded above then $b = \sup A$ (which is a real number) iff $a \leq b$ for all $a \in A$ and for every $\varepsilon > 0$ there exists $a \in A$ such that $a > b - \varepsilon$.*

It follows that if A is bounded above, $b = \sup A$, and $b \notin A$, then b is a limit point of A.

As an application of Theorem 1.11.1, we can prove

Theorem 1.11.2. *Let A and B be nonempty subsets of $\mathbb{R}$ and let $A + B = \{a + b \mid a \in A,\ b \in B\}$. Then $\sup(A + B) = \sup A + \sup B$.*

Proof. Let A and B be bounded above, then $u = \sup A$ and $v = \sup B$ are real numbers. For any $c \in A + B$ we have $c = a + b$ with $a \in A$ and $b \in B$, so $a \leq u$ and $b \leq v$, so $c \leq u + v$. And given any $\varepsilon > 0$, there are $a \in A$ such that $a > u - \varepsilon/2$ and $b \in B$ such that $b > v - \varepsilon/2$; then $a + b \in A + B$ and $a + b > (u - \varepsilon/2) + (v - \varepsilon/2) = (u + v) - \varepsilon$. By Theorem 1.11.1, $u + v = \sup(A + B)$.

If, say, A is unbounded above, then $A + B$ is also unbounded above, and $\sup A + \sup B = +\infty + \sup B = +\infty = \sup(A + B)$. ■

Let $A \subseteq \mathbb{R}$. We say $b \in \mathbb{R}$ is *a lower bound of A* if $b \leq a$ for all $a \in A$, and say that A is *bounded below* if A has a lower bound. The greatest lower bound of A is called *the infimum of A* and denoted by $\inf A$. If A is unbounded below, we write $\inf A = -\infty$.

Theorem 1.11.3. *If set $A \subseteq \mathbb{R}$ is nonempty and bounded below, then* $\inf A$ *exists.*

Proof. Let B be the set of lower bounds of A. Then B is nonempty, and every element a of A is an upper bound of B, so B is bounded above; hence, $b = \sup B$ exists. Now, every $a \in A$ is an upper bound of B, so $a \geq b$, so b is a lower bound of A; also, $b \geq c$ for every $c \in B$, so c is the greatest lower bound of A. Hence, $c = \inf A$. ■

1.12. The nested intervals principle and the base d expansion of reals

The completeness of $\mathbb{R}$ can also be expressed using *the nested intervals principle*:

Theorem 1.12.1. *Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ be a sequence of closed bounded intervals.* (Such a sequence is said to be *nested*.) *Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$, that is, there is $x \in \mathbb{R}$ such that $x \in I_n$ for all $n \in \mathbb{N}$.*

(Notice that the assertion of the theorem doesn't hold for non-closed intervals: put $I_n = (0, 1/n]$, $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} I_n = \emptyset$.)

Proof. For every n, let $I_n = [a_n, b_n]$, $a_n < b_n$. Since the intervals I_n are nested, we have that $a_1 \leq a_2 \leq \cdots$ and $b_1 \geq b_2 \geq \cdots$. I claim that for every n and m, $a_n < b_m$. Indeed, if $n \leq m$, then $a_n \leq a_m < b_m$; if $n \geq m$, then $a_n < b_n \leq b_m$. This implies that for every m, b_m is an upper bound of the set $A = \{a_n,\ n \in \mathbb{N}\}$. Let $x = \sup A$; then for every n, $x \geq a_n$ and $x \leq b_n$, so $x \in I_n$. ■

In Theorem 1.12.1, if the length of the nested intervals becomes arbitrarily small, then their intersection consists of a single point:

Theorem 1.12.2. *Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ be a sequence of closed intervals such that for any $\varepsilon > 0$ there exists n such that $|I_n| < \varepsilon$. Then $\bigcap_{n=1}^{\infty} I_n = \{x\}$ for some x.*

Proof. By the way of contradiciton, assume that $x, y \in \bigcap_{n=1}^{\infty} I_n$ and $x < y$. There is $n \in \mathbb{N}$ such that $|I_n| < y - x$, but $x, y \in I_n$, contradiction. ■

The nested intervals principle allows to construct digital expansions of real numbers. *The binary expansion* of real numbers in the interval $I = [0, 1]$ is defined as follows. I is subdivided into two subintervals of equal lengths, $I_0 = [0, 1/2]$ and $I_1 = [1/2, 1]$. Then, by induction, we define closed intervals $I_{e_1,\ldots,e_n}$ of length $1/2^n$ for all $n \in \mathbb{N}$ and all $e_1, \ldots, e_n \in \{0, 1\}$: if an interval $I_{e_1,\ldots,e_n} = [a, b]$ has already been defined, we subdivide it into two subintervals $I_{e_1,\ldots,e_n,0} = \left[a, \frac{a+b}{2}\right]$ and $I_{e_1,\ldots,e_n,1} = \left[\frac{a+b}{2}, b\right]$ of lengths $|b - a|/2$. Now, given any infinite sequence $e_1, e_2, \ldots$ with $e_i \in \{0, 1\}$ for all i, we have the sequence of nested intervals $I_{e_1} \supseteq I_{e_1,e_2} \supseteq I_{e_1,e_2,e_3} \supseteq \cdots$ of arbitrarily small lengths; for the single point x in the intersection $\bigcap_{n=1}^{\infty} I_{e_1,\ldots,e_n}$ we write $x = .e_1e_2\cdots$ and call it *the binary expansion* of x. If $x = .e_1e_2\ldots e_k000\ldots$ for some k, that is $e_n = 0$ for all $n > k$, we also write $x = .e_1e_2\ldots e_k$.

Conversely, for any point $x \in [0, 1]$ there are $e_1, e_2, \ldots \in \{0, 1\}$ such that $x \in I_{e_1,\ldots,e_n}$ for every n: if, by induction, $x \in I_{e_1,\ldots,e_n}$, then $x \in I_{e_1,\ldots,e_n,0}$ or $x \in I_{e_1,\ldots,e_n,1}$, and we put e_{n+1} to be 0 or 1 accordingly. Hence, every point $x \in [0, 1]$ has a binary expansion $.e_1e_2\cdots$. Unfortunately, there are points whose binary expansion is not unique: if, for some n and $e_1, \ldots, e_n$, x is the boundary point of the intervals $I_{e_1,\ldots,e_n,0}$ and $I_{e_1,\ldots,e_n,1}$, then $x = .e_1\cdots e_n0111\cdots$ and $x = .e_1\cdots e_n1000\cdots$. (This means that the mapping $\{.e_1e_2\cdots,\ e_i \in \{0, 1\}\} \longrightarrow [0, 1]$ from the set of binary expansions to the interval $[0, 1]$ is surjective but not injective.)

Next, we extend the binary expansion to all real numbers as follows: for $x \geq 1$, we represent the integer part $[x]$ of x in the form $[x] = 2^m + 2^{m-1}c_1 + \cdots + 2^0c_m$ for some $m \in \mathbb{N}$ and $c_1, \ldots, c_m \in \{0, 1\}$ (which representation exists and is unique), find a binary expansion $.e_1e_2\cdots$ of the fractional part $\{x\}$ of x, and write $x = 1c_1\cdots c_m.e_1e_2\cdots$. We call c_i and e_j *the digits* of x in *the binary numerical system*.

In the same way, subdividing our intervals into three equal parts instead ot two and labeling them with the symbols $0, 1, 2$, we get *the ternary* expansion of real numbers; into ten equal parts and using as digits the symbols $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ we get the (most popular) *decimal* expansion; and for any $d \in \mathbb{N} \setminus \{1\}$, subdividing our intervals into 16 equal parts and using as digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F$, we obtain *the hexadecimal* expansion; in general, for any integer $d \geq 2$, subdividing our intervals into d equal parts (and using as digits the first d symbols from the list $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, \ldots$) we get the *the base* d expansion.

1.13. Existence and uniqueness of the real numbers

An alternative way to introduce real numbers is *constructive*, where the set of real numbers is constructed and its properties are proved, not declared as axioms. I'll describe this approach very briefly, without any justification.

We start by defining *whole numbers* (nonnegative integers) $0 = \emptyset$, $1 = \{0\} = \{\emptyset\}$, $2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$, etc.: for every n already constructed we add its *successor* $n' = n \cup \{n\}$. We define addition and multiplication on these numbers inductively by $0 + n = n + 0 = n$, $m' + n = n + m' = (n + m)'$, $0n = n0 = 0$,

$m'n = nm' = nm + n$. We then add negative integers to get the set $\mathbb{Z}$ of integers, extend addition and multiplication on $\mathbb{Z}$, and check that the axioms (P1)-(P7) and (P9) hold. Since we have a set of positive integers, we can also define an order on $\mathbb{Z}$, and check that (P10)-(P12) also hold.

Next, we introduce rational numbers as follows. We consider the set $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ of ordered pairs (n, m) with $n \in \mathbb{Z}$ and $m \in \mathbb{Z} \setminus \{0\}$. We say that two ordered pairs (n_1, m_1) and (n_2, m_2) are equivalent if $n_1 m_2 = n_2 m_1$. We then define a fraction n/m as the set of all ordered pairs equivalent to the pair (n, m), and define the set $\mathbb{Q}$ of rational numbers as the set of all such fractions. We define addition and multiplication of fractions as usual, and check that they satisfy the axioms (P1)-(P8). We say that n/m is positive if both $n, m > 0$ or both < 0, and check that this set of positive numbers satisfies (P10)-(P12).

Finally, we need to complete $\mathbb{Q}$ to get $\mathbb{R}$. If $\mathbb{R}$ exists and $\mathbb{Q}$ is a subset of $\mathbb{R}$, then every real number x defines the Dedekind cut (A, B) of $\mathbb{Q}$ by $A = \{r \in \mathbb{Q} : r \leq x\}$ and $B = \{r \in \mathbb{Q} : r > x\}$, and, as $\mathbb{Q}$ is dense in $\mathbb{R}$, x is uniquely defined by this cut. Thus, *we define* the set $\mathbb{R}$ of real numbers as the set of Dedekind cuts (A, B) of $\mathbb{Q}$ with the property that B has no minimal element. (As for A, it may have a maximal element x, – and then $x \in \mathbb{Q}$ and the cut corresponds to x; and may not have it, – in which case the cut defines a "new", non-rational real number.) We then define addition and multiplication on $\mathbb{R}$ (for addition, for $x_1 = (A_1, B_1)$ and $x_2 = (A_2, B_2)$ we define $x_1 + x_2$ as $(A_1 + A_2, B_1 + B_2)$ (well, almost: if x_1 and x_2 are irratinal and $r = x_1 + x_2$ is rational we have to take $\big((A_1 + A_2) \cup \{r\}, B_1 + B_2)\big)$) and, again, check that all the axioms (P1)-(P13) are satisfied for this $\mathbb{R}$ with these operations.

On the other hand, axioms (P1)-(P13) define real numbers uniquely, in the following sense. Given two ordered fields F and K, *an isomorphism* between F and K is a bijective mapping $f\colon F \longrightarrow K$ that agrees with the operations "+", "·", and the order "<": for all $a, b \in \mathbb{R}$, $f(a+b) = f(a)+f(b)$ and $f(ab) = f(a)f(b)$, and $f(a) < f(b)$ iff $a < b$. If such an isomorphism exists, we say that F and K are *isomorphic*. Isomorphic fields are "copies" of each other, what is true for one is true for the other, – after replacing the elements of the first with the corresponding elements of the second.

The following theorem says that there is a unique, up to isomorphism, set of reals:

Theorem 1.13.1. *Any two complete ordered fields are isomorphic.*

Here is a sketch of the proof. Let $\mathbb{R}$ and $\mathbb{R}'$ be two complete ordered fields (that is, two sets with operations of addition and multiplication and order relations, satisfying axioms (P1)-(P13)). We need to construct an isomorphism $f\colon \mathbb{R} \longrightarrow \mathbb{R}'$. $\mathbb{R}$ has special elements zero 0 and one 1, and $\mathbb{R}'$ has its own zero $0'$ and $1'$; we define $f(0) = 0'$ and $f(1) = 1'$. Then we define $f\colon \mathbb{N} \cup \{0\} \longrightarrow \mathbb{N}' \cup \{0'\}$ (where $\mathbb{N}'$ is the set of natural numbers in $\mathbb{R}'$) inductively: if $f(n) = n'$, we put $f(n+1) = n' + 1'$. We then prove by induction that f is an isomorphism between $\mathbb{N} \cup \{0\}$ and $\mathbb{N}' \cup \{0'\}$, – a bijection that agrees with addition, multiplication, and order: for any n, m, $f(n+m) = f(n) + f(m)$, $f(nm) = f(n)f(m)$, and $f(n) < f(m)$ iff $n < m$. We then extend f to $\mathbb{Z}$ by $f(-n) = -f(n)$, $n \in \mathbb{N}$, and show that it is still an isomorphism. Then we extend f to $\mathbb{Q}$ by defining $f(n/m) = f(n)/f(m)$, $n, m \in \mathbb{Z}$, show that f is well defined (that $f(r)$ for $r \in \mathbb{Q}$ doesn't depend on the representation of r in the form m/n), and is an isomorphism. Finally, we extend f to $\mathbb{R}$ as follows: given $x \in \mathbb{R}$ we consider the Dedekind cut $A = \{r \in \mathbb{Q} : r \leq x\}$, $B = \{r \in \mathbb{Q} : r > x\}$, show that $f(A)$, $f(B)$ is a Dedekind cut of $\mathbb{R}'$, and define $f(x)$ to be the corresponding element of $\mathbb{R}'$, that is, $f(x) = \sup f(A)$. And, we prove that this $f\colon \mathbb{R} \longrightarrow \mathbb{R}'$ is an isomorphism.

1.14. Countable and uncountable sets

Two sets X and Y are said to have *same cardinality*, or be *equicardinal*, if there is a bijection $X \longrightarrow Y$; we write $|X| = |Y|$ in this case. We write $|X| \leq |Y|$ if $|X| = |A|$ for some $A \subseteq Y$ (that is, there is an injection $X \longrightarrow Y$). We write $|X| < |Y|$ and say that the cardinality of X is smaller than the cardinality of Y if $|X| \leq |Y|$ and $|X| \neq |Y|$.

A set X is said to be *countable* if $|X| = |\mathbb{N}|$, that is, $X = \{x_1, x_2, \ldots\}$, where all x_n are distinct. Non-countable infinite sets are said tobe *uncountable*.

It is easy to see that every infinite set X contains a countable subset (so, $|X| \geq |\mathbb{N}|$); that any subset of a countable set is at most countable (that is, countable or finite); and that any surjective image of a countable set is at most countable.

We also have:

Theorem 1.14.1. (i) *A finite union of countable sets is countable.*

(ii) *A countable union of finite sets is at most countable.*

(iii) *A Cartesian product of two countable sets is countable.*

(iv) *A countable union of countable sets is countable.*

Proof. Actually, (iv) implies (i), (ii), and (iii); but I will derive (iv) from (iii) and (iii) from (ii).

(i) Let $A_1, \dots, A_n$ be countable sets; "count" the elements in each of them: let $A_i = \{x_{i,1}, x_{i,2}, \dots\}$, $i = 1, \dots, n$. Then the sequence $x_{1,1}, x_{2,1}, \dots, x_{n,1}, x_{2,1}, x_{2,2}, \dots, x_{n,2}, x_{3,1}, \dots$ "counts" the elements of $\bigcup_{i=1}^n A_i$. More formally, construct the mapping $f\colon \mathbb{N} \longrightarrow \bigcup_{i=1}^n A_i$ in the following way. For $k \in \mathbb{N}$, let i, j be such that $k = n(i-1) + j$ where $i, j \in \mathbb{N}$, $1 \le j \le n$, and define $f(k) = x_{j,i}$. Since for any $k \in \mathbb{N}$ the pair (i, j) with this property is unique, f is well defined. Since for every pair (i, j) we have $x_{i,j} = f(k)$ for $k = n(i-1) + j$, f is surjective. So, $\left|\bigcup_{i=1}^n A_i\right| \le |\mathbb{N}|$; since $|A_1| = |\mathbb{N}|$, we have $\left|\bigcup_{i=1}^n A_i\right| = |\mathbb{N}|$.

(ii) Let $A_1, A_2, \dots$ be a sequence of finite sets; if some of them are empty remove them from this sequence, and thus assume that all $A_i \ne \emptyset$. Let $|A_i| = n_i$ and $A_i = \{x_{i,1}, \dots, x_{i,n_i}\}$, $i \in \mathbb{N}$. Define $m_0 = 0$ and for every $i \in \mathbb{N}$, $m_i = m_{i-1} + n_i$, then $0 = m_0 < m_1 < m_2 < \cdots$, and so, by induction, $m_i \ge i$ for all i. Define $f\colon \mathbb{N} \longrightarrow \bigcup_{i=1}^\infty A_i$ in the following way: for $k \in \mathbb{N}$, find i such that $m_{i-1} < k \le m_i$, and put $f(k) = x_{i,k-m_{i-1}}$. f is well defined and is surjective, since for any $i \in \mathbb{N}$ and $1 \le j \le n_i$ we have $x_{i,j} = f(m_{i-1} + j)$. So, $\left|\bigcup_{i=1}^n A_i\right| \le |\mathbb{N}|$.

(iii) Let A and B be countable, $A = \{x_1, x_2, \dots\}$ and $B = \{y_1, y_2, \dots\}$. Then $A \times B$ is representable as a countable union of finite sets: $A \times B = \bigcup_{k=2}^\infty \big\{(x_i, y_j) \bigm| i + j = k\big\}$. So, $A \times B$ is at most countable; but since $|A \times B| \ge |A \times \{y_1\}| = |A|$, $A \times B$ is countable.

(iv) Now let $A_1, A_2, \dots$ be an infinite sequence of countable sets, $A_i = \{x_{i,1}, x_{i,2}, \dots\}$, $i \in \mathbb{N}$. Define a function $f\colon \mathbb{N} \times \mathbb{N} \longrightarrow \bigcup_{i=1}^\infty A_i$ by $f(i, j) = x_{i,j}$. Then f is surjective, and $\mathbb{N} \times \mathbb{N}$ is countable by (iii), so $\bigcup_{i=1}^\infty A_i$ is at most countable; but since $\left|\bigcup_{i=1}^\infty A_i\right| \ge |A_1|$ which is infinite, it is countable. ∎

As a corollary we obtain that the set of rational unmbers is countable:

Theorem 1.14.2. $\mathbb{Q}$ *is countable.*

Proof. Every $q \in \mathbb{Q}$ has form $q = m/n$ with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$; this means that the mapping $f\colon \mathbb{Z} \times \mathbb{N} \longrightarrow \mathbb{Q}$, $f(m, n) = m/n$, is surjective. Since $\mathbb{Z}$ and $\mathbb{N}$ are countable, $\mathbb{Z} \times \mathbb{N}$ is countable, so $\mathbb{Q}$ is countable. ∎

We will need the following nice fact:

Theorem 1.14.3. *Any set of disjoint intervals in* $\mathbb{R}$ *is at most countable.*

Proof. Let $\mathcal{I}$ be a set of disjoint intervals. For every $I \in \mathcal{I}$ choose a rational number $x_I \in I$. (We can do this since $\mathbb{Q}$ is dense in $\mathbb{R}$.) Since for any $I, J \in \mathcal{I}$ we have $I \cap J = \emptyset$, $x_I \notin J$, so $x_I \ne x_J$. Hence, the mapping $\mathcal{I} \longrightarrow \mathbb{Q}$, $I \mapsto x_I$, is injective. So, $|\mathcal{I}| \le |\mathbb{Q}|$. ∎

Definition. A real number α is said to be *algebraic* if it satisfies a polynomial equation with rational coefficients: there are $n \in \mathbb{N}$ and $a_0, a_1, \dots, a_n \in \mathbb{Q}$, $a_n \ne 0$, such that $a_n\alpha^n + \cdots + a_1\alpha + a_0 = 0$. Non-algebraic real numbers are called *transcendental*.

Theorem 1.14.4. *The set* $\mathcal{A}$ *of algebraic numbers is countable.*

Proof. For every $n \in \mathbb{N}$ let $\mathcal{P}_n$ be the set of polynomials of degree n with rational coefficients, that is, functions $\mathbb{R} \longrightarrow \mathbb{R}$ of the form $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ where $a_0, a_1, \dots, a_n \in \mathbb{Q}$, $a_n \ne 0$. For each n, there is a bijection between $\mathcal{P}_n$ and $(\mathbb{Q}\setminus\{0\}) \times \mathbb{Q}^{n-1}$, $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \leftrightarrow (a_n, a_{n-1}, \dots, a_0)$; since $\mathbb{Q}^n$ is countable, $\mathcal{P}_n$ is also countable. The set $\mathcal{P}$ of all nonconstant polynomials with rational coefficients is the countable union $\bigcup_{n=1}^\infty P_n$, and so, is also countable. By definition, a real number α is algebraic iff α is a root, $f(\alpha) = 0$, of some $f \in \mathcal{P}$. For every $f \in \mathcal{P}$ let A_f be the set of roots of f; it is well known that A_f is finite. So, the set $\mathcal{A} = \bigcup_{f \in P} A_f$ of algebraic numbers is a countable union of finite sets, and so, is countable.

But not all infinite sets are countable!

Theorem 1.14.5. (Cantor) $\mathbb{R}$ *is uncountable.*

Proof. Let (a_n) be any sequence in $\mathbb{R}$; I claim that there is $a \in \mathbb{R}$ which is not in this sequence. Indeed, choose a closed interval I_1 such that $a_1 \notin I_1$. Then choose a closed subinterval I_2 of I_1 such that $a_2 \notin I_2$. Etc., by induction: if a sequence $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n$ of closed intervals has been chosen so that $a_k \notin I_k$ for $k = 1, \ldots, n$, then we choose an interval $I_{n+1} \subseteq I_n$ such that $a_{n+1} \notin I_{n+1}$. We obtain an infinite sequence $I_1 \supseteq I_2 \supseteq \cdots$ of closed intervals such that for every n, $a_n \notin I_n$. By the nested intervals principle, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$; let $a \in \bigcap_{n=1}^{\infty} I_n$. Then for any $n \in \mathbb{N}$, $a \in I_n$ and $a_n \notin I_n$, so $a \neq a_n$. ■

Usually, Theorem 1.14.5 is proved in a different way. The set of all $\{0,1\}$-sequences (that is, sequences of 0s and 1s) is denoted by $\{0,1\}^{\mathbb{N}}$, or $2^{\mathbb{N}}$.

Theorem 1.14.6. *The set* $2^{\mathbb{N}}$ *is uncountable.*

Proof. We will show that there is no surjective mapping $\mathbb{N} \longrightarrow 2^{\mathbb{N}}$, that is, that for any sequence S of elements $2^{\mathbb{N}}$ there is $s \in 2^N$ such that s is not in the range of S. Let $(s_1, s_2, \ldots)$ be a sequence of elements of $2^{\mathbb{N}}$, that is, a sequence of $\{0,1\}$-sequences. For every $n \in \mathbb{N}$, let $s_n = (e_{n,1}, e_{n,2}, \ldots)$, where $e_{n,i} \in \{0,1\}$ for all i. Define $s = (d_1, d_2, \ldots) \in 2^{\mathbb{N}}$ by $d_n = 1 - e_{n,n}$, $i \in \mathbb{N}$. Then for every $n \in \mathbb{N}$, $s \neq s_n$ since the n-th element d_n of s is not equal to the n-th element $e_{n,n}$ of s_n. ■

Next, we notice that the set $2^{\mathbb{N}}$ of $\{0,1\}$-sequences and the interval $[0,1]$ are "almost" equicardinal. Indeed, we have a surjective mapping $f\colon 2^{\mathbb{N}} \longrightarrow [0,1]$ defined by $f(e_1, e_2, \ldots) = .e_1e_2\ldots \in [0,1]$ (the binary expansion of a real number). This mapping is surjective (every real number has a binary expansion), but not quite injective: some real numbers have two distinct binary expansions. However, the set A of such real numbers is countable: $A = \{l/2^k : l \in \mathbb{Z},\ k \in \mathbb{N}\}$, the set B of correspondong binary expansions is also countable, f defines a bijection between $2^{\mathbb{N}} \setminus B$ and $[0,1] \setminus A$, so the cardinalities of $2^{\mathbb{N}}$ and of $[0,1]$ are equal.

As a corollary of Theorem 1.14.5 we obtain:

Theorem 1.14.7. *The sets of irrational and of transcendental numbers are nonempty, and, moreover, uncountable.*

Proof. Let $\mathcal{I}$ be the set of irrational numbers, $\overline{A}$ be the set of algebraic numbers, $\mathcal{T}$ be the set of transcendental numbers. If $\mathcal{I}$ were countable, then $\mathbb{R} = \mathbb{Q} \cup \mathcal{I}$ would be countable, which is false; so, $\mathcal{I}$ is uncountable. Similarly, if $\mathcal{T}$ were countable, then $\mathbb{R} = \overline{A} \cup \mathcal{T}$ would be countable, which is false; so, $\mathcal{I}$ is uncountable. ■

The cardinality of $\mathbb{R}$ is called *cardinality of the continuum*. It is easy to see that any interval in $\mathbb{R}$ also has cardinality of the continuum.

It looks natural that the cardinality of a "continuous" interval in $\mathbb{R}$ is larger than the cardinality of the dense, but not "continuous" set of rational numbers. There are however sets in $\mathbb{R}$ that seem to be very small, almost invisible, but have the cardinality of the continuum!

Such is the classical *Cantor set*, which is constructed in the following way: Subdivide the interval $[0,1]$ into three subintervals of length 1/3, let I_0 be the first first of them, $I_0 = \left[0, \frac{1}{3}\right]$, and I_1 be the third, $I_1 = \left[\frac{2}{3}, 1\right]$. Define $C_1 = I_0 \cup I_1$. Next, subdivide I_0 into three equal parts, let $I_{0,0}$ be the first of them, $I_{0,0} = \left[0, \frac{1}{9}\right]$, and $I_{0,1}$ be the third of them, $I_{0,1} = \left[\frac{2}{9}, \frac{1}{3}\right]$; do the same with I_1 to get subintervals $I_{1,0}$ and $I_{1,1}$. Put $C_2 = I_{0,0} \cup I_{0,1} \cup I_{1,0} \cup I_{1,1}$. And so on, by induction: for $n \in \mathbb{N}$, C_n is a union of 2^n disjoint closed intervals of length $1/3^n$, indexed by $0,1$-sequences of length n:

$$C_n = \bigcup_{s \in S_n} I_s, \text{where } S_n = \{0,1\}^n = \big\{(e_1, \ldots, e_n) \bigm| e_i \in \{0,1\} \text{ for all } i\big\}.$$

The set $C = \bigcap_{n=1}^{\infty} C_n$ is called *Cantor's set*.

The points of C are in a one-to-one correspondence with the set $S = 2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$ of infinite $\{0.1\}$-sequences. Indeed, given such a sequence $s = (e_1, e_2, \ldots) \in S$, we have the nested sequence $I_{e_1} \subset I_{e_1,e_2} \subset I_{e_1,e_2,e_3} \subset \cdots$ of closed intervals; by the nested intervals principle, there is a point x_s in the intersection $\bigcap_{n=1}^{\infty} I_{e_1,\ldots,e_n}$. Since the length of the intervals tends to 0 (the length of $I_{e_1,\ldots,e_n}$ is $(1/3)^n$ and for any $\varepsilon > 0$ there eixists n such that $(1/3)^n < \varepsilon$), such a point x_s is unique, so we have a mapping $f\colon S \longrightarrow C$, $f(s) = x_s$. f is surjective: for every point $x \in C$, x is contained in I_{e_1} for some $e_1 \in \{0,1\}$, then it is contained in I_{e_1,e_2} for some $e_1 \in \{0,1\}$, and for any $n \in \mathbb{N}$, c is contained in $I_{e_1,\ldots,e_n}$ for some $e_1, \ldots, e_n$; then $x = x_s$ for the sequence $s = (e_1, e_2, \ldots)$. Also, f is injective: if $s = (e_1, e_2, \ldots)$ and $t = (d_1, d_2, \ldots)$ are two distinct

$\{0,1\}$-sequences, with $e_n \neq d_n$, then $x_s \in I_{e_1,\ldots,e_n}$ and $x_t \in I_{d_1,\ldots,d_n}$, but $I_{e_1,\ldots,e_n}$ and $I_{d_1,\ldots,dn}$ are disjoint, so $x_s \neq x_t$. So, f is a bijection, and so, C has cardinality of the continuum.

At the same time, C is *nowhere dense*: it is not dense in any interval in $\mathbb{R}$. Also, C has *zero measure*: for any n, C_n is a union of 2^n intervals of length $1/3^n$, so "the total length" (the measure) $\lambda(C_n)$ of C_n is $2^n/3^n = (2/3)^n$. Since $0 < 2/3 < 1$, for every $\varepsilon > 0$ there exists n such that $\lambda(C_n) < \varepsilon$. Since $C \subseteq C_n$ for every n, we obtain that "the length" (the measure) $\lambda(C) < \varepsilon$ for every $\varepsilon > 0$, so, $\lambda(C) = 0$. So, C is a nowhere dense set of measure zero with cardinality of the continuum!

In fact, it is easy to describe C with the help of ternary (that is, base 3) expansions of real numbers. Every number $x \in [0,1]$ has a ternary expansion $0.c_1c_2\cdots$ with $c_n \in \{0,1,2\}$, and C is the set of all numbers from $[0,1]$ whose ternary expansion contains no 1s (but only 0s and 2s).

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2. Sequences and their limits

2.1. Converging and diverging sequences

A sequence $(x_n) = (x_n)_{n=1}^{\infty} = (x_1, x_2, \ldots)$ of real numbers is a mapping $\mathbb{N} \longrightarrow \mathbb{R}$, $n \mapsto x_n$.

A sequence is not the set of its values! The sets $\{0,1,0,1,0,\ldots\}$, $\{1,0,1,0,1,\ldots\}$, $\{0,1,1,1,1,\ldots\}$ are all equal (to the set $\{0,1\}$) but the sequences $(0,1,0,1,0,\ldots)$, $(1,0,1,0,1,\ldots)$, $(0,1,1,1,1,\ldots)$ are all distinct: two sequences (x_n) and (y_n) are equal IFF $x_n = y_n$ for all n.

Informally, we see a sequence as a process with discrete time: it "jumps" from point x_1 to x_2, then to x_3, x_4, etc. This way, a sequence may have different behaviour: it may hit a single point: $(a,a,a,a,\ldots)$; oscillate between two points: $(a,b,a,b,\ldots)$; go to $+\infty$: $(1,2,3,4,\ldots)$; go to ∞ "oscillating" between $+$ and $-$: $(1,-2,3,-4,\ldots)$; run between $+\infty$ and $-\infty$ and back: $(0,1,0,-1,0,1,2,1,0,-1,-2,\ldots)$; being dense in an interval, or in $\mathbb{R}$, running over all rational points in this interval or in $\mathbb{R}$ respectively; etc.

We are however mostly interested in so-called "converging" sequences. We say that a sequence (x_n) *converges to* $a \in \mathbb{R}$, or *tends to* a, or *has limit* a, and that a *is the limit of* (x_n), and write $x_n \longrightarrow a$, or $\lim_{n\to\infty} x_n = a$, or $\lim x_n = a$, iff for any $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ for all $n \geq k$.

Given a point $a \in \mathbb{R}$ and $\varepsilon > 0$, the interval $(a-\varepsilon, a+\varepsilon)$ is called *the ε-neighborhood* of a. We have $\lim_{n\to\infty} x_n = a$ if for any $\varepsilon > 0$ "all but finitely many elements of sequence (x_n) are in the ε-neighborhood of a", or, in other words, "x_n are in the ε-neiborhood of a for all n large enough".

Examples. (i) Let $x_n = 1/n$, $n \in \mathbb{N}$. I claim that $\lim_{n\to\infty} x_n = 0$. Let $\varepsilon > 0$; find $k \in \mathbb{N}$ such that $k > 1/\varepsilon$, then for any $n \geq k$ we have $0 < 1/n < 1/k < \varepsilon$. Hence, for any $n \geq k$, $|x_n - 0| < \varepsilon$.
(ii) Let now $x_n = 1/\sqrt{n}$, $n \in \mathbb{N}$. I claim that $\lim_{n\to\infty} x_n = 0$. Let $\varepsilon > 0$; find $k \in \mathbb{N}$ such that $k > 1/\varepsilon^2$, then for any $n \geq k$ we have $0 < 1/\sqrt{n} < 1/\sqrt{k} < \varepsilon$. Hence, for any $n \geq k$, $|x_n - 0| < \varepsilon$.

It is important to know that a limit of a sequence, if exists, is unique, and we can call it "the limit":

Theorem 2.1.1. *If a sequence converges, its limit is unique.*

Proof. Let (x_n) be a converging sequence, and, by the way of contradiction, assume that $a = \lim_{n\to\infty} x_n$ and $b = \lim_{n\to\infty} x_n$ with $a \neq b$. W.l.o.g. assume that $a < b$. Put $\varepsilon = (b-a)/2$. Find k_1 such that $|x_n - a| < \varepsilon$ for all $n \geq k_1$ and find k_2 such that $|x_n - b| < \varepsilon$ for all $n \geq k_2$. Choose any $n \geq k_1, k_2$, then, by the triangle inequality, $|b-a| \leq |x_n - b| + |x_n - a| < \varepsilon + \varepsilon = b - a$, which is false. ■

The symbols we use in the definition of limit are not important, of course; the sentence "for any $H > 0$ there exists v such that for any $\beta > v$, $|x_\beta - a| < H$" also states that $a = \lim x_n$. (Note, however, that, traditionally, to denote integers the letters i, j, k, l, m, n, and sometimes d or r, are used; this simplifies reading of math texts.) What is really important in this definition is the order of the quantifiers: the sentence "there exist $\varepsilon > 0$ and $k \in \mathbb{N}$ such that for all $n \geq k$ we have $|x_n - a| < \varepsilon$" states that the sequence (x_n) is bounded; the sentence "there exists k such that for any $\varepsilon > 0$ for any $n \geq k$ we have $|x_n - a| < \varepsilon$" states that the sequence (x_n) is eventually constant: there exists k such that $x_n = a$ for all $n > k$.

If a sequence (x_n) doesn't converge, we say that it *diverges*. We say that it *diverges to* $+\infty$ and write $x_n \longrightarrow +\infty$, or $\lim_{n\to\infty} x_n = +\infty$, or $\lim x_n = +\infty$, if for any $M \in \mathbb{R}$ there exists k such that $x_n > M$ for all $n \geq k$. We say that it *diverges to* $-\infty$ and write $x_n \longrightarrow -\infty$, or $\lim_{n\to\infty} x_n = -\infty$, or $\lim x_n = -\infty$, if for any $M \in \mathbb{R}$ there exists k such that $x_n < M$ for all $n \geq k$. We say that it *diverges to* ∞ if $\lim |x_n| = +\infty$. (Note that a sequence may diverge, but not to $\pm\infty$ or ∞.)

2.2. Properties of converging sequences and of their limits

First of all,

Theorem 2.2.1. *A sequence (x_n) converges to $a \in \mathbb{R}$ iff $\lim |x_n - a| = 0$. In particular, $\lim x_n = 0$ iff $\lim |x_n| = 0$.*

Proof. By definition, $\lim |x_n - a| = 0$ if for any $\varepsilon > 0$ there exists k such that for all $n \geq k$, $\big||x_n - a| - 0\big| < \varepsilon$. But $\big||x_n - a| - 0\big| = |x_n - a|$, so this is equivlent to $\lim x_n = a$. ∎

A sequence (x_n) is said to be *bounded above* if there is M such that $x_n \leq M$ for all n; *bounded below* if there is N such that $x_n \geq N$ for all n; and *bounded* if it is bounded both above and below. Equivalently, (x_n) is bounded if there is M such that $|x_n| \leq M$ for all n.

Theorem 2.2.2. *If a sequence converges it is bounded.*

Proof. Let (x_n) be a converging sequence, let $\lim x_n = a$. Let k be such that $|x_n - a| < 1$ for all $n \geq k$, then $a - 1 < x_n < a + 1$ for all $n \geq k$. Let $M = \max\{x_1, \dots, x_{n-1}, a+1\}$ and $N = \min\{x_1, \dots, x_{k-1}, a-1\}$; then or any n, if $n \leq k - 1$ then $N \leq x_n \leq M$; if $n \geq k$, then $N \leq a - 1 < x_n < a + 1 \leq M$. Hence, for all n, $N \leq x_n \leq M$. ∎

The converse is not true: a sequence can be bounded but diverging. However, the following is true:

Theorem 2.2.3. *Let a sequence (x_n) converge, let $\lim x_n = a$, and let $b \in \mathbb{R}$.*
(i) *If $x_n \leq b$ for infinitely many n, then $a \leq b$.*
(ii) *If $x_n \geq b$ for infinitely many n, then $a \geq b$.*
(iii) *If $a < b$ then $x_n < b$ for all n large enough (that is, there is k such that $x_n < a$ for all $n \geq k$).*
(iv) *If $a > b$ then $x_n > b$ for all n large enough.*

Notice that if $x_n < b$ for all n, it doesn't follow that $a < b$, but only that $a \leq b$.

Proof. (iii) If $a < b$, put $\varepsilon = b - a$, then $\varepsilon > 0$, so there exists k such that for all $n \geq k$, $x_n < a + \varepsilon = b$. (So, $x_n \geq b$ for finitely many n only.)
(iv) If $a > b$, put $\varepsilon = a - b$, then $\varepsilon > 0$, so there exists k such that for all $n \geq k$, $x_n > a - \varepsilon = b$. (So, $x_n \leq b$ for finitely many n only.)

(i) and (ii) are the contrapositives of (iv) and (iii) respectively: "$x_n \leq b$ for infintely many n" means that it is not true that $x_n > b$ for all n large enough, and then, by (iv), it is not true that $a > b$, hence $a \leq b$; similarly, if $x_n \geq b$ for infinitely many n, then $a \geq b$. ∎

The squeeze theorem for sequences is the following fact:

Theorem 2.2.4. *Let (x_n), (y_n), (z_n) be sequences such that for all n, $y_n \leq x_n \leq z_n$ or $z_n \leq x_n \leq y_n$, and such that $\lim y_n = \lim z_n = a$. Then (x_n) also converges to a, $\lim x_n = a$.*

Proof. Let $\varepsilon > 0$. Find k_1 such that for all $n \geq k_1$, $|y_n - a| < \varepsilon$, so that $a - \varepsilon < y_n < a + \varepsilon$, and find k_2 such that for all $n \geq k_2$, $|z_n - a| < \varepsilon$, so that $a - \varepsilon < z_n < a + \varepsilon$. Let $k = \max\{k_1, k_2\}$. Then for any $n \geq k$ we have

$$a - \varepsilon < y_n \leq x_n \leq z_n < a + \varepsilon$$

or

$$a - \varepsilon < z_n \leq x_n \leq y_n < a + \varepsilon,$$

so $a - \varepsilon < x_n < a + \varepsilon$, so $|x_n - a| < \varepsilon$. ∎

Next, we have theorems "on arithmetic" of limits:

Theorem 2.2.5. *Let (x_n) and (y_n) be two converging sequences, let $\lim x_n = a$ and $\lim y_n = b$. Then*
(i) *$\lim(x_n + y_n) = a + b$ (that is, the sequence $(x_n + y_n)$ converges and its limit is $a + b$);*
(ii) *for any $c \in \mathbb{R}$, $\lim(cx_n) = ca$;*

(iii) $\lim(x_n y_n) = ab$;
(iv) *if* $a \neq 0$ *then* $\lim(1/x_n) = 1/a$;
(v) *if* $a \neq 0$ *then* $\lim(y_n/x_n) = b/a$.

In (iv) an (v) we should additionally assume that $x_n \neq 0$ for all n, since other wise $1/x_n$ doesn't exist. However, if $\lim x_n \neq 0$, then $x_n \neq 0$ for all but, maybe, finitely many n, which may be just ignored when we compute limits.

Proof. (i) Let $\varepsilon > 0$. Let k_1 be such $|x_n - a| < \varepsilon/2$ for all $n \geq k_1$ and k_2 be such $|y_n - b| < \varepsilon/2$ for all $n \geq k_2$, then for all $n \geq \max\{k_1, k_2\}$ we have

$$\big|(x_n + y_n) - (a + b)\big| = \big|(x_n - a) + (y_n - b)\big| \leq |x_n - a| + |y_n - b| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

(ii) follows from (iii) if we put $y_n = c$ for all $n \in \mathbb{N}$, but let's prove it independently. If $c = 0$ then (cx_n) is the constant sequence equal to 0, and $\lim cx_n = 0 = ca$. So, assume that $c \neq 0$. Let $\varepsilon > 0$. Find k such that $|x_n - a| < \varepsilon/|c|$ for all $n \geq k$. Then for any $n \geq k$,

$$|cx_n - ca| = |c| \cdot |x_n - a| < |c| \cdot \varepsilon/|c| = \varepsilon.$$

(iii) Let $\varepsilon > 0$. Since (x_n) converges, it is bounded; let $M > 0$ be such that $|x_n| \leq M$ for all n. Let k_1 be such $|x_n - a| < \varepsilon/(2|b| + 1)$ for all $n \geq k_1$ and k_2 be such $|y_n - b| < \varepsilon/(2M)$ for all $n \geq k_2$, then for all $n \geq \max\{k_1, k_2\}$ we have

$$\begin{aligned}\big|x_n y_n - ab\big| = \big|x_n y_n - x_n b + x_n b - ab\big| &\leq |x_n y_n - x_n b| + |x_n b - ab| = |x_n| \cdot |y_n - b| + |x_n - a| \cdot |b| \\ &< M\varepsilon/(2M) + \varepsilon|b|/(2|b| + 1) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

(iv) Let $a \neq 0$, let $\varepsilon > 0$. Assume that $a > 0$. Find k_1 such that $|x_n - a| < |a|/2$ for all $n \geq k_1$, then $|x_n| \geq |a| - |x_n - a| > |a| - |a|/2 = |a|/2$ for $n \geq k_1$. Find k_2 such that $|x_n - a| < \varepsilon|a|^2/2$ for all $n \geq k_2$. Then for any $n \geq \max\{k_1, k_2\}$ we have

$$\Big|\frac{1}{x_n} - \frac{1}{a}\Big| = \Big|\frac{a - x_n}{x_n a}\Big| = \frac{|x_n - a|}{|x_n| \cdot |a|} < \frac{\varepsilon|a|^2/2}{(|a|/2)|a|} = \varepsilon.$$

(v) follows from (iii) and (iv). ∎

The arithmetic of infinites ($1/0 = \infty$, $1/\infty = 0$, $\pm\infty + a = \pm\infty$ for any $a \in \mathbb{R}$, $(\pm\infty)a = \pm\infty$ for any $a > 0$ and $(\pm\infty)a = \mp\infty$ for any $a < 0$, $(+\infty) + (+\infty) = +\infty$, $(-\infty) + (-\infty) = -\infty$) also applies to limits:

Theorem 2.2.6. *Let (x_n) and (y_n) be sequences.*
(i) *If* $\lim x_n = 0$ *and* $x_n \neq 0$ *for all* n, *then* $\lim(1/x_n) = \infty$.
(ii) *If* $\lim x_n = \infty$, *then* $\lim(1/x_n) = 0$.
(iii) *If* $\lim x_n = \pm\infty$ *or* ∞ *and* (y_n) *converges, or is just bounded, then* $\lim(x_n + y_n) = \pm\infty$ *or* ∞ *respectively.*
(iv) *If* $\lim x_n = \pm\infty$ *or* ∞ *and* $\lim y_n > 0$, *then* $\lim(x_n y_n) = \pm\infty$ *or* ∞ *respectively. If* $\lim x_n = \pm\infty$ *or* ∞ *and* $\lim y_n < 0$, *then* $\lim(x_n y_n) = \mp\infty$ *or* ∞ *respectively.*
(v) *If* $\lim x_n = \lim y_n = \pm\infty$, *then* $\lim(x_n + y_n) = \pm\infty$.

Proof. (i) Let $M > 0$. Find k such that $|x_n| < 1/M$ for all $n \geq k$. Then $|1/x_n| > M$ for all $n \geq k$.
(ii) Let $\varepsilon > 0$. Find k such that $|x_n| > 1/\varepsilon$ for all $n \geq k$. Then $|1/x_n| < \varepsilon$ for all $n \geq k$.
(iii) Let $\lim x_n = +\infty$ and assume that (y_n) is bounded below: let N be such that $y_n \geq N$ for all n. Given $M \in \mathbb{R}$, find k such that $x_n > M - N$ for all $n \geq k$, then $x_n + y_n > (M - N) + N = M$ for all $n \geq k$. The other cases can be proved similarly, or just deduced from this one.
(iv) Let $\lim x_n = +\infty$ and $\lim y_n = a > 0$; the other cases are similar. Find k_1 such that $y_n > a/2$ for all $n \geq k_1$. Given $M \in \mathbb{R}$, find k_2 such that $x_n > 2M/a$ for all $n \geq k_2$. Then for any $n \geq \max\{k_1, k_2\}$, $|x_n y_n| > (2M/a)(a/2) = M$.

(v) Let $\lim x_n = \lim y_n = +\infty$. Given $M \in \mathbb{R}$, find k such that $x_n > M$ and $y_n > 0$ for all $n \geq k$, then $x_n + y_n > M$ for all such n. ■

A version of the squeeze theorem also exists for infinite limits is called *a comparison principle*:

Theorem 2.2.7. *Let (x_n) and (y_n) be sequences. If $\lim x_n = +\infty$ and $y_n \geq x_n$ for all n, then $\lim y_n = +\infty$. If $\lim x_n = -\infty$ and $y_n \leq x_n$ for all n, then $\lim y_n = -\infty$.*

Proof. Let $\lim x_n = +\infty$ and let $y_n \geq x_n$ for all n. Let $M \in \mathbb{R}$, let k be such that $x_n > M$ for all $n \geq k$, then also $y_n > M$ for all $n \geq k$. ■

2.3. Some standard limits

• By the Archimedean property of $\mathbb{N}$, $\lim_{n\to\infty} n = +\infty$; it follows that $\lim(1/n) = 0$, and by induction, for any $d \in \mathbb{N}$, $\lim n^d = +\infty$ and $\lim(1/n^d) = 0$.
• Let $a \in \mathbb{R}$; consider the sequence (a^n):
If $a > 1$, put $x = a - 1$, then by Bernoulli's inequality $a^n = (1+x)^n \geq 1 + nx$, so $\lim a^n = +\infty$ (by Theorem 2.2.7).
If $a = 1$, then $\lim a^n = 1$.
If $0 \leq a < 1$, then $a^{-1} > 1$, so $\lim (a^n)^{-1} = \lim (a^{-1})^n = +\infty$, so $\lim a^n = 0$.
If $-1 < a < 0$, then $\lim |a^n| = \lim |a|^n = 0$, so $\lim a^n = 0$.
If $a = -1$, $\lim a^n$ doesn't exist (the sequence $(-1)^n = (-1, 1, -1, 1, \dots)$ diverges).
If $a < -1$, then $\lim |a^n| = \lim |a|^n = +\infty$, so $\lim a^n = \infty$. (Since the sign of a^n switches from $+$ to $-$ and back, $\lim a^n \neq +\infty$ or $-\infty$ in this case.)
• It is easy to compute limits in situations where Theorems 2.2.5 and 2.2.6 apply, like $\lim \frac{(2/n+3)(2+(1/2)^n)}{5-10/n^3} = 6/5$.

There are, however, so-called *indeterminate* limits, which may be of types $0/0$ (that is, x_n/y_n with both $\lim x_n = \lim y_n = 0$), ∞/∞, $0 \cdot \infty$, $\infty - \infty$, etc. Dealing with such limits requires some creativity.
• Here is an example of finding an indeterminate limit of the form $\infty - \infty$:

$$\lim\bigl(\sqrt{n+1} - \sqrt{n}\bigr) = \lim \frac{\bigl(\sqrt{n+1} - \sqrt{n}\bigr)\bigl(\sqrt{n+1} + \sqrt{n}\bigr)}{\sqrt{n+1} + \sqrt{n}} = \lim \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \lim \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0.$$

• Sequences of the form $a_d n^d + a_{d-1} n^{d-1} + \cdots + a_1 n + a_0$ where $d \in \mathbb{N}$, $a_0, \dots, a_d \in \mathbb{R}$, $a_d \neq 0$, are called *polynomial of degree* d; every polynomial sequnce diverges to ∞. If x_n and y_n are polynomial sequences, the sequence x_n/y_n is called *rational*. Limits of rational sequences are indeterminate of type ∞/∞, but can be easily found: let $x_n = a_d n^d + a_{d-1} n^{d-1} + \cdots + a_1 n + a_0$ and $y_n = b_c n^c + b_{c-1} n^{c-1} + \cdots + b_1 n + b_0$, then

$$\frac{x_n}{y_n} = \frac{a_d n^d + a_{d-1} n^{d-1} + \cdots + a_1 n + a_0}{b_c n^c + b_{c-1} n^{c-1} + \cdots + b_1 n + b_0} = \frac{a_d + a_{d-1} n^{-1} + \cdots + a_1 n^{-d+1} + a_0 n^{-d}}{b_c + b_{c-1} n^{-1} + \cdots + b_1 n^{-c+1} + b_0 n^{-c}} n^{d-c}.$$

The limit of the fraction is equal to $a_d/b_c \neq 0$ and everything depends on the limit of n^{d-c}: if $d = c$ it is 1 and $\lim(x_n/y_n) = a_d/b_c$; if $d > c$ it is ∞ and $\lim(x_n/y_n) = \infty$; if $d < c$ it is 0 and $\lim(x_n/y_n) = 0$.
• For any $d \in \mathbb{N}$ and $a > 1$, $n^d/a^n \longrightarrow 0$. Indeed, put $x = a - 1$, then $x > 0$. For any $d \in \mathbb{N}$ and $n > d$, it follows from the binomial formula that $a^n = (1+x)^n > \binom{n}{d+1} x^{d+1} = \frac{n(n-1)\cdots(n-d)}{d!} x^{d+1}$. Here $x^{d+1}/d!$ is a constant, and $n(n-1)\cdots(n-d)$ is a polynomial sequence of degree $d+1$, so $\lim n^d / \bigl(\binom{n}{d+1} x^{d+1}\bigr) = 0$, so $\lim n^d/a^n = 0$ by the squeeze theorem.
• For any $a > 1$, $a^n/n! \longrightarrow 0$. Indeed, choose any $k \in \mathbb{N}$ with $k > 2a$, then for any $n \geq k$ we have $n! = k!(k+1)(k+2)\cdots n > k! k^{n-k} = \frac{k!}{k^k} k^n$, so $a^n/n! < \frac{k^k}{k!} a^n/k^n = \frac{k^k}{k!}(a/k)^n < \frac{k^k}{k!}(1/2)^n$. Since $(1/2)^n \longrightarrow 0$, $\frac{a^n}{n!} \longrightarrow 0$ as well by the squeeze theorem.
• You can also show that $n!/n^n \longrightarrow 0$.

For sequences (x_n) and (y_n) with $\lim x_n = \lim y_n = \infty$ let's write $x_n \prec y_n$ if $\lim(x_n/y_n) = 0$. Clearly, $n^d \prec n^c$ for any $d, c \in \mathbb{N}$ with $d < c$, and $a^n \prec b^n$ for any $b > a > 1$. We also have that $n^d \prec a^n \prec n! \prec n^n$ for all $d \in \mathbb{N}$ and all $a > 1$.
• One more "standard limit" is $\lim \sqrt[n]{a}$ for $a > 0$. ($\sqrt[n]{a}$ is defined as $b > 0$ such that $b^n = a$; we haven't proved its existence yet, but we could do this in the same way as we did for $n = 2$.) We may assume that

$a > 1$ (if $a < 1$, we can replace it by a^{-1}). Then $\sqrt[n]{a} > 1$ for all n (since if $b < 1$, then $b^n < 1$). For any $\varepsilon > 0$, since $(1+\varepsilon)^n \longrightarrow \infty$, there exists k such that $(1+\varepsilon)^n > a$ for all $n \geq k$, then $1+\varepsilon > \sqrt[n]{a} > 1$ for all $n \geq k$, so $0 < \sqrt[n]{a} - 1 < \varepsilon$, so $\lim \sqrt[n]{a} = 1$.
• The next one is $\lim \sqrt[n]{n}$; it is also equal to 1. Let $\varepsilon > 0$; since $(1+\varepsilon)^n/n \longrightarrow \infty$, there exists k such that $(1+\varepsilon)^n > n$ for all $n \geq k$, then $1+\varepsilon > \sqrt[n]{n} > 1$ for all $n \geq k$, so $\lim \sqrt[n]{n} = 1$.

2.4. Monotone sequences

There are situations where we can prove the existence of a limit without guessing it. One of them is when the sequence is *monotone*.

Let (x_n) be a sequence in $\mathbb{R}$. It is said to be *increasing* if $x_{n+1} \geq x_n$ for all n; *strictly increasing* if $x_{n+1} > x_n$ for all n; *decreasing* if $x_{n+1} \leq x_n$ for all n; and *strictly decreasing* if $x_{n+1} < x_n$ for all n. (In some books, *nondecreasing*, *increasing*, *nonincreasing*, and *decreasing* respectively.) (x_n) is said to be *monotone* if it is increasing or decreasing, and *strictly monotone* if it is strictly increasing or strictly decreasing.

Every monotone sequence always has a limit (which can be infinite however):

Theorem 2.4.1. *Let (x_n) be a monotone sequence. If (x_n) is increasing and bounded above, then* $\lim x_n = \sup\{x_n,\ n \in \mathbb{N}\}$*; if it is increasing and unbounded above, then* $\lim x_n = +\infty$*; if it is decreasing and bounded below, then* $\lim x_n = \inf\{x_n,\ n \in \mathbb{N}\}$*; if it is decreasing and unbounded below, then* $\lim x_n = -\infty$.

Proof. Let (x_n) be increasing and bounded above, let $a = \sup\{x_n,\ n \in \mathbb{N}\}$. Then $x_n \leq a$ for all n. For any $\varepsilon > 0$ there exists k such that $x_k > a - \varepsilon$; then for any $n \geq k$ we have $x_n \geq x_k > a - \varepsilon$; hence, $|x_n - a| < \varepsilon$ for all $n \geq k$. So, $\lim x_n = a$.

Now let (x_n) be increasing and unbounded above. Let $M \in \mathbb{R}$; there exists k such that $x_k > M$, and then for any $n \geq k$, $x_n \geq x_k > M$. This proves that $\lim x_n = +\infty$.

The case of decreasing sequences can be treated similarly, or can be reduced to the case of increasing sequences by replacing x_n by $-x_n$. ∎

As a corollary, we have the following *comparison principle*:

Theorem 2.4.2. *Let (x_n) and (y_n) be increasing sequences and let $x_n \leq y_n$ for all n. If (y_n) converges, then (x_n) also converges, and* $\lim x_n \leq \lim y_n$.

(Sure, a similar theorem holds for decreasing sequences.)

Proof. If (y_n) converges, it is bounded above, thus (x_n) is also bounded above, and so, converges. And we have $\lim x_n = \sup\{x_n,\ n \in \mathbb{N}\} \leq \sup\{y_n,\ n \in \mathbb{N}\} = \lim y_n$. ∎

Examples. (i) Let $a > 0$, $a \neq 1$ and $x_n = 1 + a + a^2 + \cdots + a^n = \sum_{i=0}^n a^i$, $n \in \mathbb{N}$. Then (x_n) is a strictly increasing sequence. As can be proved by induction, for any n, $x_n = \frac{a^{n+1}-1}{a-1}$; if $a > 1$, $\lim x_n = +\infty$, if $0 < a < 1$, $\lim x_n = \frac{1}{1-a}$.
(ii) Let $y_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} = \sum_{i=0}^n \frac{1}{i!}$. (y_n) is also an increasing sequence, and for any n, $y_n \leq 1 + x_n$ for $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^n}$. Since (x_n) converges, with $\lim x_n = 2$, (y_n) also converges with $2.5 < \lim y_n \leq 1 + \lim x_n = 3$. (Actually, $\lim y_n = e$.)
(iii) Let $z_n = \left(1 + \frac{1}{n}\right)^n$, $n \in \mathbb{N}$. (This sequence has indeterminate limit of type 1^∞.) I claim that (z_n) is increasing and $z_n \leq y_n = \sum_{i=0}^n \frac{1}{i!}$ for all n; this will prove that (z_n) converges, with $\lim z_n \leq \lim y_n \leq 3$. Indeed, using the binomial formula, for any n, we have:

$$\begin{aligned} z_n &= (1+1/n)^n \\ &= 1 + n\tfrac{1}{n} + \tbinom{n}{2}\tfrac{1}{n^2} + \tbinom{n}{3}\tfrac{1}{n^3} + \cdots + \tbinom{n}{n}\tfrac{1}{n^n} = 1 + 1 + \tfrac{1}{2!}\tfrac{n(n-1)}{n^2} + \tfrac{1}{3!}\tfrac{n(n-1)(n-2)}{n^3} + \cdots + \tfrac{1}{n!}\tfrac{n(n-1)(n-2)\cdots 1}{n^n} \\ &= 1 + 1 + \tfrac{1}{2!}\left(1 - \tfrac{1}{n}\right) + \tfrac{1}{3!}\left(1 - \tfrac{1}{n}\right)\left(1 - \tfrac{2}{n}\right) + \cdots + \tfrac{1}{n!}\left(1 - \tfrac{1}{n}\right)\left(1 - \tfrac{2}{n}\right)\cdots\left(1 - \tfrac{n-1}{n}\right). \end{aligned}$$

This shows that $z_n \leq y_n$ for all n. If we compare the expression for z_n with

$$\begin{aligned} z_{n+1} = 1 + 1 + \tfrac{1}{2!}\left(1 - \tfrac{1}{n+1}\right) + \tfrac{1}{3!}\left(1 - \tfrac{1}{n+1}\right)\left(1 - \tfrac{2}{n+1}\right) + \cdots + \tfrac{1}{n!}\left(1 - \tfrac{1}{n+1}\right)\left(1 - \tfrac{2}{n+1}\right)\cdots\left(1 - \tfrac{n-1}{n+1}\right) \\ + \tfrac{1}{(n+1)!}\left(1 - \tfrac{1}{n+1}\right)\left(1 - \tfrac{2}{n+1}\right)\cdots\left(1 - \tfrac{n}{n+1}\right) \end{aligned}$$

we see that every summand in the expression for z_{n+1} is larger (or equal) than the corresponding summand in the expression for z_n, plus we have one more additional, last summand. Hence, $z_{n+1} > z_n$.

The limit of (z_n) is a transcendental number, called "e" (*Euler's number*).

(iv) Let $a > 0$; define a sequence (v_n) inductively by $v_1 = 1$ and $v_{n+1} = \frac{1}{2}(v_n + a/v_n)$ for all n. I claim that (v_n) is decreasing and bounded below by $\sqrt{a}$ starting with $n = 2$. Indeed, for any $n \geq 2$, by the arithmetic-geometric mean inequality, $v_n = \frac{1}{2}(v_{n-1} + a/v_{n-1}) \geq \sqrt{v_{n-1}(a/v_{n-1})} = \sqrt{a}$, so $a/v_n \leq \sqrt{a} \leq v_n$, so $v_{n+1} = \frac{1}{2}(v_n + a/v_n) \leq \frac{1}{2}(v_n + v_n) = v_n$. Hence, (v_n) converges. Let $b = \lim v_n$; then $b = \lim v_{n+1} = \lim \frac{1}{2}(v_n + a/v_n) = \frac{1}{2}(b + a/b)$, so $b = a/b$, so $b = \sqrt{a}$ (since $b > 0$).

The sequence (v_n) converges to $\sqrt{a}$ fast enough, and is often used to calculate it. This method of finding $\sqrt{a}$ is called *Newton's algorithm.*

2.5. Cauchy's criterion of convergence

A sequence (x_n) is said to be *Cauchy* if for any $\varepsilon > 0$ there exists k such that $|x_n - x_m| < \varepsilon$ for all $n, m \geq k$.

The following fact is called *the Cauchy criterion of convergence*:

Theorem 2.5.1. *A sequence converges iff it is Cauchy.*

Proof. Let (x_n) be a converging sequence, let $\lim x_n = a$. Let $\varepsilon > 0$. Find k such that $|x_n - a| > \varepsilon/2$ for all $n \geq k$. Then for any $n, m \geq k$, $|x_n - x_m| \leq |x_n - a| + |x_m - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. So, (x_n) is Cauchy.

Conversely, let (x_n) be Cauchy. There exists k_1 such that $|x_n - x_m| < 1$ for all $n \geq k_1$; put $I_1 = [x_{k_1} - 1, x_{k_1} + 1]$, then $x_n \in I_1$ for all $n \geq k_1$. There exists $k_2 \geq k_1$ such that $|x_n - x_m| < 1/2$ for all $n \geq k_2$; put $I_2 = [x_{k_2} - 1/2, x_{k_2} + 1/2] \cap I_1$, then $x_n \in I_2$ for all $n \geq k_2$. And so on, by induction: we construct a nested sequence $I_1 \supseteq I_2 \supseteq \cdots$ of closed intervals with $|I_i| \leq 2/i$ for all i and a sequence $k_1 \leq k_2 \leq \cdots$ of integers such that for any i for any $n \geq k_i$, $x_n \in I_i$. By the nested intervals principle, there exists $a \in \bigcap_{i=1}^{\infty} I_i$.

I claim that $a = \lim x_n$. Indeed, let $\varepsilon > 0$; let $i > 2/\varepsilon$, so that $2/i < \varepsilon$. Then for all $n \geq k_i$ we have $x_n \in I_i$ and $a \in I_i$, so $|x_n - a| \leq |I_i| \leq 2/i < \varepsilon$. ■

A sequence (x_n) is Cauchy, and so, converges, if $x_n - x_m \longrightarrow 0$ as $n, m \to \infty$; the condition $x_{n+1} - x_n \longrightarrow 0$ is not sufficient. (The sequence $0, 1, 1/2, 0, 1/3, 2/3, 1, 3/4, 2/4, 1/4, 0, 1/5, \ldots$ is an example: it satisfies $x_{n+1} - x_n \longrightarrow 0$ but diverges.) However, if the difference $x_{n+1} - x_n$ tends to zero fast enough (namely, exponentially), then the sequence is Cauchy:

Theorem 2.5.2. *Let (x_n) be a sequence such that for some $c < 1$ and $M \in \mathbb{R}$ we have $|x_{n+1} - x_n| \leq c^n M$ for all n. Then (x_n) is Cauchy.*

Proof. Let $n, m \in \mathbb{N}$, $n < m$. Then

$$\begin{aligned}
|x_n - x_m| &= \big|x_n - x_{n+1} + x_{n+1} - x_{n+2} + \cdots + x_{m-1} - x_m\big| \\
&\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \leq c^n M + c^{n+1} M + \cdots + c^{m-1} M \\
&= c^n M(1 + c + \cdots + c^{m-n-1}) = c^n M \tfrac{1-c^{m-n}}{1-c} < c^n \tfrac{M}{1-c}.
\end{aligned}$$

Now, given $\varepsilon > 0$, find k such that $c^k < \varepsilon(1-c)/M$, then for any $n \geq k$, $|x_n - x_m| < c^n \frac{M}{1-c} \leq c^k \frac{M}{1-c} < \varepsilon$. ■

Here is an example: let $e_1, e_2, \ldots \in \{-1, 1\}$ be any sequence of "signs", and let $x_n = e_1 + \frac{e_2}{2!} + \frac{e_3}{3!} + \cdots + \frac{e_n}{n!}$. Then for any $n \geq 2$, $|x_{n+1} - x_n| = \frac{1}{(n+1)!} \leq \frac{1}{2^n}$, so (x_n) is Cauchy.

The condition of Theorem 2.5.2 is satisfied in the following situation:

Theorem 2.5.3. *Let (x_n) be a sequence such that for some $c < 1$, $|x_{n+2} - x_{n+1}| \leq c|x_{n+1} - x_n|$ for all n. Then (x_n) is Cauchy.*

Proof. We have $|x_3 - x_2| \leq c|x_2 - x_1|$, then $|x_4 - x_3| \leq c|x_3 - x_2| \leq c^2|x_2 - x_1|$, and by induction, $|x_{n+1} - x_n| \leq c^{n-2}|x_2 - x_1| = c^n|x_{n+1} - x_n|/c^2$ for all n. By Theorem 2.5.2, (x_n) is Cauchy. ■

Here is an example. Define sequence (x_n) inductively by $x_1 = 1$ and $x_{n+1} = 1 + 1/x_n$ for all n. By induction, $x_n > 0$ for all n; so, $x_n \geq 1$ for all n; so $x_n = 1 + 1/x_{n-1} \leq 1 + 1 = 2$ for all $n \geq 2$; so $x_n = 1 + 1/x_{n-1} \geq 3/2$ for all $n \geq 3$. Now, for any $n \geq 2$ we have

$$|x_{n+2} - x_{n+1}| = \big|\big(1 + \tfrac{1}{x_{n+1}}\big) - \big(1 + \tfrac{1}{x_n}\big)\big| = \big|\tfrac{1}{x_{n+1}} - \tfrac{1}{x_n}\big| = \big|\tfrac{x_n - x_{n+1}}{x_{n+1}x_n}\big| = \tfrac{|x_{n+1} - x_n|}{|x_n||x_{n+1}|} \leq \tfrac{2}{3}|x_{n+1} - x_n|.$$

So, (x_n) converges. If $a = \lim x_n$, we have $a = \lim x_{n+1} = \lim\big(1 + \frac{1}{x_n}\big) = 1 + \frac{1}{a}$, so $a^2 - a - 1 = 0$, so $a = \frac{1+\sqrt{5}}{2}$.

2.6. Limit points and subsequences

Let (x_n) be a sequence in $\mathbb{R}$. A number $a \in \mathbb{R}$ is said to be *a limit point* of (x_n) if for any $\varepsilon > 0$, the set $\{n : |x_n - a| < \varepsilon\}$ is infinite, that is, the ε-neighborhood of a contains infinitely many points of the sequence; in quantifiers: for any $\varepsilon > 0$ for any k there exists $n \geq k$ such that $|x_n - a| < \varepsilon$.

Examples.

(i) If a sequence converges, it has a unique limit point, which is its limit.
(ii) If a sequence diverges to ∞, it has no limit points.
(iii) The sequence $x_n = (-1)^n + 1/n$, $n \in \mathbb{N}$, has exactly two limit points, -1 and 1. There are sequences with three, four, or any finite set of limit points.
(iv) The sequence $\left(0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, 0, \frac{1}{4}, \ldots\right)$ is dense in $[0,1]$, thus every point of $[0,1]$ is a limit point of this sequence.
(v) The sequence of all rational numbers is dense in $\mathbb{R}$, thus every real number is a limit point of this sequence.

The fundamental *Bolzano-Weierstrass theorem* says that every bounded sequence in $\mathbb{R}$ has a limit point. (This means that closed bounded intervals in $\mathbb{R}$ are *compact*.)

Theorem 2.6.1. *Every bounded sequence in $\mathbb{R}$ has a limit point.*

Proof. Let (x_n) be a bounded sequence, let $N, M \in \mathbb{R}$ be such that $N \leq x_n \leq M$ for all n. Define $I_1 = [N, M]$. Subdivide I_1 into two closed subintervals of length $|I_1|/2$ and define I_2 to be one of these subintervals that contains infinitely many elements of (x_n). Subdivide I_2 into two closed subintervals of length $|I_2|/2 = |I_1|/4$ and define I_3 to be one of these subintervals that contains infinitely many elements of (x_n). Continuing this way, we choose, by induction, a nested sequence $I_1 \supseteq I_2 \supseteq \cdots$ of closed intervals with $|I_i| \longrightarrow 0$ as $i \longrightarrow \infty$; let $a \in \bigcap_{i=1}^{\infty} I_i$. Let $\varepsilon > 0$; find k such that $|I_k| < \varepsilon$, then $I_k \subseteq (a - \varepsilon, a + \varepsilon)$. Since I_k contains infinitely many elements of (x_n), so does the ε-neighborhood $(a - \varepsilon, a + \varepsilon)$ of a. Hence, a is a limit point of (x_n). ■

If (x_n) is a sequences and $n_1 < n_2 < n_3 < \cdots$ is a strictly increasing sequence in $\mathbb{N}$, the sequence $(x_{n_i}) = (x_{n_1}, x_{n_2}, \ldots)$ is said to be *a subsequence* of (x_n). A subsequence is indexed by "the second" index: $(x_{n_i}) = (y_i)$ where $y_i = x_{n_i}$ for all i.

Clearly, if a sequence converges, every its subsequence converges to the same limit:

Theorem 2.6.2. *If (x_n) is a sequence with $\lim x_n = a$, then for any its subsequence (x_{n_i}), $\lim x_{n_i} = a$ as well.*

Proof. I'll prove this for $a \in \mathbb{R}$ only. Let $\varepsilon > 0$; find k such that $|x_n - a| < \varepsilon$ for all $n \geq k$ and find j such that $n_j \geq k$. (This is possible since, clearly, $n_j \geq j$ for all j, so $j = k$ works.) Then for any $i \geq j$ we have $n_i \geq n_j \geq k$, so $|x_{n_i} - a| < \varepsilon$. ■

The converse is not true, of course: a diverging sequence may have converging subsequences. If a subsequence (x_{n_i}) of (x_n) converges, its limit $\lim_{i \to \infty} x_{n_i}$ is called *a subsequential limit* of (x_n). Actually, the subsequential limits of a sequence are just its limit points:

Theorem 2.6.3. *For any sequence (x_n), a point $a \in \mathbb{R}$ is a limit point of (x_n) iff a is a subsequential limit of (x_n).*

Proof. Let a be a limit point of (x_n). Find $n_1 \in \mathbb{N}$ such that $|x_{n_1} - a| < 1$. Find $n_2 > n_1$ such that $|x_{n_2} - a| < 1/2$. (It exists since the set of n for which $|x_n - a| < 1/2$ is infinite.) Find $n_3 > n_2$ such that $|x_{n_3} - a| < 1/3$. Etc., by induction: we construct a sequence $n_1 < n_2 < n_3 < \cdots$ of integers such that $|x_{n_i} - a| < 1/i$ for all i. Since $1/i \longrightarrow 0$ as $i \longrightarrow \infty$, by the squeeze theorem $x_{n_i} \longrightarrow a$ as $i \longrightarrow \infty$.

Conversely, let (x_{n_i}) be a converging subsequence of (x_n), let $a = \lim_{i \to \infty} x_{n_i}$. Then every neighborhood of a contains infinitely many (actually, almost all) elements of (x_{n_i}), so infinitely many elements of (x_n). So, a is a limit point of (x_n). (In quantifiers: let $\varepsilon > 0$ and let $k \in \mathbb{N}$. Find j such that $|x_{n_i} - a| < \varepsilon$ for all $i \geq j$. Find $i \geq j$ such that $n_i \geq k$. Then $|x_{n_i} - a| < \varepsilon$ and $n_i \geq k$.) ■

We can now reformulate the Bolzano-Weierstrass theorem:

Theorem 2.6.4. *Every bounded sequence in $\mathbb{R}$ has a converging subsequence.*

And we can give another proof of it, based on the following nice lemma:

Lemma 2.6.5. *Every sequence in $\mathbb{R}$ has a monotone subsequence.*

Proof. Let (x_n) be a sequence in $\mathbb{R}$ (or, actually, in any totally ordered set). Let's say that $m \in \mathbb{N}$ is *a peak point* (a peak point index?) of (x_n) if $x_m \geq x_n$ for all $n > m$. There are two cases: If there are infinitely many peak points $m_1 < m_2 < m_3 < \cdots$, then $x_{m_1} \geq x_{m_2} \geq x_{m_3} \geq \cdots$, so, (x_{m_i}) is a decreasing subsequence of (x_n). If there are only finitely many peak points, let m be the maximal one; then for any $n > m$ there is $k > n$ such that $x_k > x_n$. We then can construct a (strictly) increasing subsequence of (x_n): take $n_1 = m+1$, then find $n_2 > n_1$ such that $x_{n_2} > x_{n_1}$, then find $n_3 > n_2$ such that $x_{n_3} > x_{n_2}$, etc. ■

Now, the Bolzano-Weierstrass theorem can be proved as follows: given a bounded sequence, it has an (also bounded) monotone subsequence, which converges.

Let's also prove that if a bounded sequence has a unique limit point, then it converges to this point. Indeed, let a be the only limit point of a bounded sequence (x_n). Let $\varepsilon > 0$; if there are infinitely many elements of (x_n) outside of $I = (a - \varepsilon, a + \varepsilon)$ (that is, infinitely many n such that $|x_n - a| \geq \varepsilon$), they form a subsequence of (x_n) outside of I, which has a limit point outside of I, contradiction.

2.7. Limsup and liminf

Let (x_n) be a sequence. *The limit superior* or *the upper limit* of (x_n), $\limsup_{n\to\infty} x_n$ or $\overline{\lim}_{n\to\infty} x_n$, is defined as $\lim_{k\to\infty} \sup\{x_n \mid n \geq k\}$. *The limit inferior* or *the lower limit* of (x_n), $\liminf_{n\to\infty} x_n$ or $\underline{\lim}_{n\to\infty} x_n$, is defined as $\lim_{k\to\infty} \inf\{x_n \mid n \geq k\}$. Clearly, $\liminf x_n \leq \limsup x_n$.

Let's only discuss $\limsup$, $\liminf$ is the dual notion. We define $s_1 = \sup\{x_1, x_2, x_3, \ldots\}$, $s_2 = \sup\{x_2, x_3, x_4, \ldots\}$, $s_3 = \sup\{x_3, x_4, x_5, \ldots\}$, etc. Then $s_1 \geq s_2 \geq s_3 \geq \cdots$, that is, (s_k) is a decreasing sequence; thus, it has a limit, $\inf\{s_k,\ k \in \mathbb{N}\}$, which may be a real number (if (x_n) doesn't diverge to $-\infty$), or $-\infty$ (if $(x_n) \longrightarrow -\infty$). It also can be that $s_k = +\infty$ for all k (if (x_n) is unbounded above); in this case we put $\limsup x_n = +\infty$.

The good thing about limsup and liminf is that they always exist! (although may be infinite).

Examples. In all the examples below, $\liminf x_n = -1$ and $\limsup x_n = 1$, but the sequences demonstrate different behavior:

(i) $x_n = (-1)^n(1 + 1/n)$, $n \in \mathbb{N}$. The sequence has infinitely many elements greater than 1, and the subsequence of these elements converges to 1; the sequence has infinitely many elements less than -1, and the subsequence of these elements converges to -1.

(ii) $x_n = (-1)^n(1 - 1/n)$, $n \in \mathbb{N}$. No elements of the sequence are greater than 1, and there is a subsequence that converges to 1 from below; no elements of the sequence are less than -1, and there is a subsequence that converges to -1 from above.

(iii) $x_n = (-1)^n + 1/n$, $n \in \mathbb{N}$. The sequence has infinitely many elements greater than 1, and the subsequence of these elements converges to 1; no elements of the sequence are less than 1, and there is a subsequence that converges to -1 from above.

(iv) Let (x_n) be a sequence of all rational numbers in $[-1, 1]$. Then the situation is the same as in (ii), but the set of limit points is the whole interval $[-1, 1]$ this time.

$\limsup$ and $\liminf$ can be characterized by the following property:

Theorem 2.7.1. *For a sequence (x_n) we have $s = \limsup x_n \in \mathbb{R}$ iff for any $\varepsilon > 0$ there exists k such that for all $n \geq k$, $x_n < s + \varepsilon$ and for any k there exists $n \geq k$ such that $x_n > s - \varepsilon$; we have $r = \liminf x_n \in \mathbb{R}$ iff for any $\varepsilon > 0$ there exists k such that for all $n \geq k$, $x_n > s - \varepsilon$ and for any k there exists $n \geq k$ such that $x_n < s + \varepsilon$.*

Proof. I'll prove this for $\limsup$ *only*. Let $s = \limsup x_n$, that is, $s = \lim s_k$ where $s_k = \sup\{x_n : n \geq k\}$, $k \in \mathbb{N}$. Let $\varepsilon > 0$. Find k such that $s_k < s + \varepsilon$, then for all $n \geq k$ we have $x_n \leq s_k < s + \varepsilon$. On the other hand, since the sequence (s_k) is decreasing, we have $s_k \geq s$ for all k, and there is $n \geq k$ such that $x_n > s_k - \varepsilon \geq s - \varepsilon$.

Conversely, let s be such that for any $\varepsilon > 0$ (i) there exists k such that for any $n \geq k$, $x_n < s + \varepsilon$, and

(ii) for any k there exists $n \geq k$ such that $x_n > s - \varepsilon$. Then for any $\varepsilon > 0$, by (ii), for any k, $s_k > s - \varepsilon$, so $s_k \geq s$; and by (i), there exists k such that $s_k < s + \varepsilon$; so, $s = \inf\{s_k,\ k \in \mathbb{N}\} = \lim s_k = \limsup x_n$. ■

It follows that for a sequence (x_n), $\limsup x_n$ (if finite) is the maximal limit point of (x_n) and $\liminf x_n$ (if finite) is the minimum limit point of (x_n). (This is why they are also called "the upper" and "the lower" limits.)

Here is one more proof of the Bolzano-Weierstrass theorem: if a sequence is bounded, its limsup is a limit point.

As a corollary, we get:

Theorem 2.7.2. *A sequence (x_n) converges with $\lim x_n = a$ iff $\liminf x_n = \limsup x_n = a$.*

This fact remains true for the infinite limits: $\lim x_n = \pm\infty$ iff $\limsup x_n = \liminf x_n = \pm\infty$.

The lim sup (and, similarly, the lim inf) can also be characterized as follows:

Theorem 2.7.3. *Let (x_n) be a sequence, let $A = \big\{a \in \mathbb{R} :$ the set $\{n : x_n > a\}$ is infinite$\big\}$ and let $B = \big\{a \in \mathbb{R} :$ the set $\{n : x_n > a\}$ is finite$\big\}$. We then have $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in a$ and $b \in B$; if both A and B are nonempty then $\limsup x_n = \sup A = \inf B$; if $A = \emptyset$ then $\limsup x_n = -\infty$; if $B = \emptyset$ then $\limsup x_n = +\infty$.*

Proof. Clearly, $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in A$ and $b \in B$. Assume that both $A, B \neq \emptyset$. Let $s = \limsup x_n$. For any $b > s$, put $\varepsilon = b - s$, then there exists k such that $x_n < s + \varepsilon = b$ for all $n \geq k$, that is, the set $\{n : x_n > a\}$ is finite, and $b \in B$. For any $a < s$, put $\varepsilon = s - a$, then for any k there is $n \geq k$ such that $x_n > s - \varepsilon = a$, that is, the set $\{n : x_n > a\}$ is infinite, and $a \in A$. So, s is the common supremum of A and infimum of B.

If $A = \emptyset$, then $\lim x_n = -\infty$ by definition. If $B = \emptyset$, then (x_n) is unbounded above, and $\limsup x_n = +\infty$. ■

Clearly, if (x_n) and (y_n) are two sequences with $x_n \leq y_n$ for all n, then $\liminf x_n \leq \liminf y_n$ and $\limsup x_n \leq \limsup y_n$. The arithmetical properties of limsup and liminf are not as good as those of limits:

Theorem 2.7.4. *For any sequences (x_n) and (y_n), $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$, $\liminf(x_n + y_n) \geq \liminf x_n + \liminf y_n$ and if $x_n, y_n \geq 0$, then also $\limsup(x_n y_n) \leq \limsup x_n \limsup y_n$ and $\liminf(x_n y_n) \geq \liminf x_n \liminf y_n$.*

Proof. For every $k \in \mathbb{N}$ let $s_k = \sup\{x_n : n \geq k\}$ and $r_k = \sup\{y_n : n \geq k\}$. Then for any k for any $n \geq k$ we have $x_n + y_n \leq s_k + r_k$, so $t_k = \sup\{x_n + y_n : n \geq k\} \leq s_k + r_k$. Hence,

$$\limsup(x_n + y_n) = \lim t_k \leq \lim(s_k + r_k) = \lim s_k + \lim r_k = \limsup x_n + \limsup y_n.$$

If $x_n, y_n \geq 0$ for all n. then for any k for any $n \geq k$ we have $x_n y_n \leq s_k r_k$, so $v_k = \sup\{x_n y_n : n \geq k\} \leq s_k r_k$. Hence,

$$\limsup(x_n y_n) = \lim v_k \leq \lim(s_k r_k) = \lim s_k \lim r_k = \limsup x_n \limsup y_n.$$

For lim inf the proof is similar. ■

Here is an example where $\limsup(x_n + y_n) < \limsup x_n + \limsup y_n$: $(x_n) = (1, -1, 1, -1, 1, -1, \ldots)$ and $(y_n) = (-1, 1, -1, 1, -1, 1, \ldots)$. We have $\limsup x_n = \limsup y_n = 1$ whereas $\lim(x_n + y_n) = 0$.

As an example of application of limsups and liminfs, let's prove *Stolz's theorem*:

Theorem 2.7.5. *Let (x_n) be a sequence and (y_n) be a strictly increasing sequence diverging to $+\infty$. Then $\limsup \frac{x_n}{y_n} \leq \limsup \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$, and $\liminf \frac{x_n}{y_n} \geq \liminf \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$. If $a = \lim \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$ exists, then $\lim \frac{x_n}{y_n} = a$ as well.*

Proof. Let $b > \limsup \frac{x_{n+1}-x_n}{y_{n+1}-y_n}$. Find k such that $\frac{x_{n+1}-x_n}{y_{n+1}-y_n} < b$ for all $n \geq k$. Since $y_{n+1} - y_n > 0$ for all n, this implies that $x_{n+1} - x_n < b(y_{n+1} - y_n)$ for all $n \geq k$. For all $n > k$ we then have

$$x_n = x_k + (x_{k+1} - x_k) + \cdots + (x_n - x_{n-1}) \leq x_k + b(y_{k+1} - y_k) + \cdots + b(y_n - y_{n-1}) = x_k - by_k + by_n.$$

For all n large enough we have $y_n > 0$, so for all n large enough and $> k$ we have $\frac{x_n}{y_n} \leq \frac{x_k - by_k}{y_n} + b$. Taking limsup of both parts, we get

$$\limsup \tfrac{x_n}{y_n} \leq \limsup_{n\to\infty} \tfrac{x_k - by_k}{y_n} + b = \lim_{n\to\infty} \tfrac{x_k - by_k}{y_n} + b = 0 + b = b.$$

Since this is true for all $b > \limsup \frac{x_{n+1}-x_n}{y_{n+1}-y_n}$, we obtain that $\limsup \frac{x_n}{y_n} \leq \limsup \frac{x_{n+1}-x_n}{y_{n+1}-y_n}$.

The proof of $\liminf \frac{x_n}{y_n} \geq \liminf \frac{x_{n+1}-x_n}{y_{n+1}-y_n}$ is similar. If $\frac{x_{n+1}-x_n}{y_{n+1}-y_n} \longrightarrow a$, then $\liminf \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = \limsup \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = a$, so $a \leq \liminf \frac{x_n}{y_n} \leq \limsup \frac{x_n}{y_n} \leq a$, so $\liminf \frac{x_n}{y_n} = \limsup \frac{x_n}{y_n} = a$, so $\lim \frac{x_n}{y_n} = a$. ■

Here is another theorem of this sort:

Theorem 2.7.6. *Let (x_n) be a sequence with $x_n > 0$ for all n. Then $\limsup \sqrt[n]{x_1 \cdots x_n} \leq \limsup x_n$, $\liminf \sqrt[n]{x_1 \cdots x_n} \geq \liminf x_n$, and if $\lim x_n$ exists, then $\lim \sqrt[n]{x_1 \cdots x_n} = \lim x_n$.*

Proof. Let $b > \limsup x_n$; find k such that $x_n < b$ for all $n \geq k$. Then for any $n > k$,

$$x_1 \cdots x_n = (x_1 \cdots x_k)(x_{k+1} \cdots x_n) < (x_1 \cdots x_k) b^{n-k} = \big(x_1 \cdots x_k / b^k\big) b^n,$$

so $\sqrt[n]{x_1 \cdots x_n} \leq \sqrt[n]{c} \cdot b$ where $c = x_1 \cdots x_k / b^k$. As $n \longrightarrow \infty$, $\lim \sqrt[n]{c} = 1$, so $\limsup \sqrt[n]{x_1 \cdots x_n} \leq 1 \cdot b = b$. Since this is true for all $b > \limsup x_n$, we obtain that $\limsup \sqrt[n]{x_1 \cdots x_n} \leq \limsup x_n$.

A similar proof shows that $\liminf \sqrt[n]{x_1 \cdots x_n} \geq \liminf x_n$. If (x_n) has a limit a, then $\liminf x_n = \limsup x_n = a$, so $a \leq \liminf \sqrt[n]{x_1 \cdots x_n} \leq \limsup \sqrt[n]{x_1 \cdots x_n} \leq a$, so $\liminf \sqrt[n]{x_1 \cdots x_n} = \limsup \sqrt[n]{x_1 \cdots x_n} = a$, so $\lim \sqrt[n]{x_1 \cdots x_n} = a$. ■

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3. Limits of functions

3.1. Limits of functions

Recall that, given a set A of real numbers, a number a is said to be *a limit point* of A if for any $\varepsilon > 0$ there exists $x \in A \setminus \{a\}$ such that $|x - a| < \varepsilon$ (that is, any neighborhood of a contains a point of A distinct from a).

Theorem 3.1.1. *a is a limit point of a set $A \subseteq \mathbb{R}$ iff there is a sequence (x_n) in $A \setminus \{a\}$ such that $x_n \longrightarrow a$.*

Proof. Let a be a limit point of A. For every $n \in \mathbb{N}$ find a point $x_n \in A \setminus \{a\}$ such that $|x_n - a| < 1/n$. Then (x_n) is a sequence in $A \setminus \{a\}$ that converges to a.

Conversely, let (x_n) be a sequence in $A \setminus \{a\}$ that converges to a. Then for any $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $|x_k - a| < \varepsilon$, and $x_k \in A \setminus \{a\}$. Hence, a is a limit point of A. ■

Now, let $A \subseteq \mathbb{R}$, let $f\colon A \longrightarrow \mathbb{R}$ be a function, and let a be a limit point of A. We say that f *has limit b at a*, or $f(x)$ *tends to b as $x \longrightarrow a$*, and write $\lim_{x\to a} f(x) = b$ or $f(x) \longrightarrow b$ as $x \longrightarrow a$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in A \setminus \{a\}$ satisfying $|x - a| < \delta$ we have $|f(x) - b| < \varepsilon$.

Examples.
(i) Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a constant function, $f(x) = b$ for all $x \in \mathbb{R}$. Then for any $a \in \mathbb{R}$, $\lim_{x\to a} f(x) = b$. Indeed, given any $\varepsilon > 0$, any $\delta > 0$ works: take, for instance, $\delta = 100$, then for any x with $|x - a| < \delta$ we have $|f(x) - b| = 0 < \varepsilon$.
(ii) Let $f(x) = 2x$, $x \in \mathbb{R}$. Then for any $a \in \mathbb{R}$, $\lim_{x\to a} f(x) = 2a$. Indeed, for any $\varepsilon > 0$ put $\delta = \varepsilon/2$, then for any x with $|x - a| < \delta$ we have $|f(x) - 2a| = |2x - 2a| = 2|x - a| < \varepsilon$.

(iii) Let $f(x)=x^2$, $x\in\mathbb{R}$. I claim that $\lim_{x\to 2} f(x)=4$. For any x we have $|f(x)-4|=|x^2-4|=|x-2|\cdot|x+2|$; and if $|x-2|<1$ then $1<x<3$, so $|x+2|<5$. Let $\varepsilon>0$, put $\delta=\min\{1,\varepsilon/5\}$; then for any x with $|x-2|<\delta$ we have $|f(x)-4|<(\varepsilon/5)\cdot 5=\varepsilon$.

(iv) Again, let $f(x)=x^2$, $x\in\mathbb{R}$. I claim that $\lim_{x\to 3} f(x)=9$. For any x we have $|f(x)-9|=|x^2-9|=|x-3|\cdot|x+3|$; and if $|x-3|<1$ then $2<x<4$, so $|x+2|<6$. Let $\varepsilon>0$, put $\delta=\min\{1,\varepsilon/6\}$; then for any x with $|x-3|<\delta$ we have $|f(x)-3|<(\varepsilon/6)\cdot 6=\varepsilon$. (Notice that for $f(x)=x^2$, in the definition of $\lim_{x\to a} f(x)$, δ depends not only on ε, but also on a.)

A function $f\colon A\longrightarrow\mathbb{R}$ *doesn't have* limit b at a limit point a of A, $f(x)\not\longrightarrow b$ as $x\longrightarrow a$, if there is some $\varepsilon>0$ such that for every $\delta>0$ there exists a point $x\in A\setminus\{a\}$ such that $|x-a|<\delta$ but $|f(x)-b|\geq\varepsilon$; in other words, if, for some $\varepsilon>0$, a is a limit point of the set $\big\{x\in A : |f(x)-b|\geq\varepsilon\big\}$.

Limits of functions can also be defined via sequences:

Theorem (the sequential definition of limits). *Let $f\colon A\longrightarrow\mathbb{R}$ be a function and a be a limit point of A. Then $\lim_{x\to a} f(x)=b$ iff for every sequence (x_n) in $A\setminus\{a\}$ with $x_n\longrightarrow a$ one has $f(x_n)\longrightarrow b$.*

Proof. Let $\lim_{x\to a} f(x)=b$. Let (x_n) be a sequence in $A\setminus\{a\}$ with $x_n\longrightarrow a$. Let $\varepsilon>0$. Find $\delta>0$ such that for any $x\in A\setminus\{a\}$ with $|x-a|<\delta$ one has $|f(x)-b|<\varepsilon$. Find k such that $|x_n-a|<\delta$ for all $n\geq k$. Then $|f(x_n)-b|<\varepsilon$ for all $n\geq k$. This proves that $f(x_n)\longrightarrow b$.

Now assume that $f(x)\not\longrightarrow b$ as $x\longrightarrow a$. Then there exists $\varepsilon>0$ such that for any $\delta>0$ there is a point $x\in A\setminus\{a\}$ with $|x-a|<\delta$ such that $|f(x)-b|\geq\varepsilon$. This means that a is a limit point of the set $B=\big\{x\in A\setminus\{a\}\ \big|\ |f(x)-b|\geq\varepsilon\big\}$. Hence, there exists a sequence (x_n) in $B\subseteq A\setminus\{a\}$ that converges to a, and by the definition of B, $f(x_n)\not\longrightarrow b$. ■

3.2. Properties of limits of functions

Properties of limits of functions are similar to those of limits of sequences (and can be deduced from each other).

A limit of a function, if exists, is unique:

Theorem 3.2.1. *Let $f\colon A\longrightarrow\mathbb{R}$ be a function and let a be a limit point of A; if $\lim_{x\to a} f(x)=b$ and $\lim_{x\to a} f(x)=c$, then $b=c$.*

Proof. Assume, w.l.o.g., that $b>c$, let $\varepsilon=(b-c)/2$. Find $x\in A\setminus\{a\}$ such that $|f(x)-b|<\varepsilon/2$ and $|f(x)-c|<\varepsilon/2$, then $|b-c|\leq|f(x)-b|+|f(x)-c|<\varepsilon/2+\varepsilon/2=\varepsilon=b-c$, contradiction. ■

Theorem 3.2.2. *Let $f\colon A\longrightarrow\mathbb{R}$ be a function and let a be a limit point of A; then $\lim_{x\to a} f(x)=b$ iff $\lim_{x\to a}|f(x)-b|=0$. In particular, $\lim_{x\to a} f(x)=0$ iff $\lim_{x\to a}|f(x)|=0$.*

This follows from the identity $\big||f(x)-b|-0\big|=|f(x)-b|$.

We say that a function f is *bounded* on a subset B of $\mathrm{Dom}(f)$ if $f(B)$ is a bounded, that is, there exists M such that $|f(x)|\leq M$ for all $x\in B$. Respectively, we say that f is *bounded above* on B if $f(B)$ is bounded above, and that f is *bounded below* on B if $f(B)$ is bounded below.

Theorem 3.2.3. *Let $f\colon A\longrightarrow\mathbb{R}$ be a function and let a be a limit point of A. If a finite $\lim_{x\to a} f(x)=b\in\mathbb{R}$ exists, then f is bounded in a neighborhood of a.*

Proof. Find $\delta>0$ such that $|f(x)-b|<1$ for all $x\in A\setminus\{a\}$ with $|x-a|<\delta$. Put $M=|b|+1$ if $a\notin A$ and $M=\max\{|b|+1,|f(a)\}$ if $a\in A$; then for any $x\in A$ with $|x-a|<\delta$ we have $|f(x)|\leq M$. ■

Theorem 3.2.4. *Let $f\colon A\longrightarrow\mathbb{R}$ be a function, let a be a limit point of A, let $\lim_{x\to a} f(x)=b\in\mathbb{R}$, and let $c\in\mathbb{R}$. If $f(x)\leq c$ for all $x\in A\setminus\{a\}$ in a neighborhood of a, then $b\leq c$; if $b>c$, then $f(x)>c$ for all $x\in A\setminus\{a\}$ in a neighborhood of a. Similarly, if $f(x)\geq c$ for all $x\in A\setminus\{a\}$ in a neighborhood of a, then $b\geq c$; if $b<c$, then $f(x)<c$ for all $x\in A\setminus\{a\}$ in a neighborhood of a.*

Proof. I'll only prove that if $b>c$, then $f(x)>c$ in a neighborhood of a. Put $\varepsilon=b-c$, find $\delta>0$ such that $|f(x)-b|<\varepsilon$ for all $x\in A\setminus\{a\}$ with $|x-a|<\delta$, then $f(x)>b-\varepsilon=c$ for all such x. ■

Here is the squeeze theorem:

Theorem 3.2.5. *Let $f, g, h\colon A \longrightarrow \mathbb{R}$ be functions, let a be a limit point of A, let $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = b$, and let $g(x) \le f(x) \le h(x)$ or $h(x) \le f(x) \le g(x)$ for all $x \in A \setminus \{a\}$ in a neighborhood of a. Then $\lim_{x\to a} f(x) = b$.*

Proof. Let $\varepsilon > 0$; find $\delta > 0$ such that both $|g(x) - b|, |h(x) - b| < \varepsilon$ and $g(x) \le f(x) \le h(x)$ or $g(x) \le f(x) \le h(x)$ for all $x \in A \setminus \{a\}$ with $|x - a| < \delta$. Then $b - \varepsilon < g(x) \le f(x) \le h(x) < b + \varepsilon$ or $b - \varepsilon < h(x) \le f(x) \le g(x) < b + \varepsilon$, so $|f(x) - b| < \varepsilon$ for all such x. ■

We can actually prove this theorem using the sequential definition of limits and the squeeze theorem for sequences:

Proof. Let (x_n) be any sequence in $A \setminus \{a\}$ such that $x_n \longrightarrow a$. Then for all n large enough we have $g(x_n) \le f(x_n) \le h(x_n)$ or $g(x_n) \le f(x_n) \le h(x_n)$. ($n$ must be large so that x_n is in the neighborhood of a where these inequalities hold.) Since both $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} h(x_n) = b$, by the squeeze theorem for sequences, $\lim_{n\to\infty} f(x_n) = b$. So, $\lim_{x\to a} f(x) = b$. ■

Let $f, g\colon A \longrightarrow \mathbb{R}$ be two functions with common domain A, let $c \in \mathbb{R}$. Then $f + g, cf, fg$ are also functions $A \longrightarrow \mathbb{R}$, defined by $(f + g)(x) = f(x) + g(x)$, $(cf)(x) = cf(x)$, and $(fg)(x) = f(x)g(x)$ for all $x \in A$. The function g/f, defined by $(g/f)(x) = g(x)/f(x)$, has domain $A \setminus \{x : f(x) = 0\}$.

Theorem 3.2.6. *Let $f, g\colon A \longrightarrow \mathbb{R}$ be functions, let a be a limit point of A, and let $\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(x) = c$. Then $\lim_{x\to a}(f(x) + g(x)) = b + c$, $\lim_{x\to a}(f(x)g(x)) = bc$, and if $b \ne 0$, $\lim_{x\to a} g(x)/f(x) = c/b$.*

I'll give two proofs. Here is "an ε-δ" proof:

Proof. Let $\varepsilon > 0$. Find $\delta > 0$ such that both $|f(x)-b|, |g(x)-c| < \varepsilon/2$ whenever $x \in A\setminus\{a\}$ and $|x-a| < \delta$. (Well, in more details: let $\delta_1 > 0$ such that both $|f(x) - b| < \varepsilon/2$ whenever $x \in A \setminus \{a\}$ and $|x - a| < \delta_1$, let $\delta_2 > 0$ such that both $|g(x) - b| < \varepsilon/2$ whenever $x \in A \setminus \{a\}$ and $|x - a| < \delta_2$, put $\delta = \min\{\delta_1, \delta_2\}$.) Then $\big|(f(x) + g(x)) - (b + c)\big| \le |f(x) - b| + |g(x) - c| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for all $x \in A \setminus \{a\}$ with $|x - a| < \delta$.

Let $\varepsilon > 0$. Find $\delta > 0$ such that $|f(x) - b| < \frac{\varepsilon}{2(|c|+1)}$, $|g(x) - c| < \frac{\varepsilon}{2(|b|+1)}$, and $|f(x) - b| < 1$ for all $x \in A \setminus \{a\}$ with $|x - a| < \delta$. For such x,

$$\begin{aligned}\big|f(x)g(x) - bc\big| &= \big|f(x)g(x) - f(x)c + f(x)c - bc\big| \le \big|f(x)g(x) - f(x)c\big| + \big|f(x)c - bc\big| \\ &= |f(x)| \cdot |g(x) - c| + |f(x) - b| \cdot |c| < (|b| + 1)\tfrac{\varepsilon}{2(|b|+1)} + \tfrac{\varepsilon}{2(|c|+1)}|c| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Let $b \ne 0$; let's prove that $\lim_{x\to a} 1/f(x) = 1/b$, then it will follow that $\lim_{x\to a} g(x)/f(x) = c/b$. Let $\varepsilon > 0$. Find $\delta > 0$ such that $|f(x) - b| < \varepsilon|b|^2/2$ and $|f(x) - b| < |b|/2$, so $|f(x)| > |b|/2$, for all $x \in A \setminus \{a\}$ with $|x - a| < \delta$; then, in particular, $1/f$ is defined for such x. And for such x,

$$\Big|\frac{1}{f(x)} - \frac{1}{b}\Big| = \frac{|f(x) - b|}{|f(x)| \cdot |b|} < \frac{\varepsilon|b|^2/2}{|b|^2/2} = \varepsilon.$$

■

Now, here is "a sequential" proof:

Proof. Let (x_n) be a sequence in $A \setminus \{a\}$ with $x_n \longrightarrow a$. Then $f(x_n) \longrightarrow b$ and $g(x_n) \longrightarrow c$. Then $f(x_n)+g(x_n) \longrightarrow b+c$, $f(x_n)g(x_n) \longrightarrow bc$, and, if $b \ne 0$, $g(x_n)/f(x_n) \longrightarrow c/b$. Hence, $f(x)+g(x) \longrightarrow b+c$, $f(x)g(x) \longrightarrow bc$, $g(x)/f(x) \longrightarrow c/b$. ■

As a corollary of Theorems 3.2.4 and 3.2.6 we can now get:

Theorem 3.2.7. *Let $f, g\colon A \longrightarrow \mathbb{R}$ be two functions, let a be a limit point of A, and suppose that $f(x) \le g(x)$ for all $x \in A \setminus \{a\}$ in a neighborhood of a and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. Then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.*

Proof. Define $h(x) = g(x) - f(x)$, $x \in A$; then $h(x) \ge 0$ for all $x \in A \setminus \{a\}$ in a neighborhood of a, so $\lim_{x\to a} g(x) - \lim_{x\to a} f(x) = \lim_{x\to a} h(x) \ge 0$. ■

This next theorem is "new", we didn't have it for sequences:

Theorem 3.2.8. *Let $f\colon A \longrightarrow \mathbb{R}$ and $g\colon B \longrightarrow \mathbb{R}$ be functions with $f(A) \subseteq B$, let a be a limit point of A and b be a limit point of B, let $\lim_{x\to a} f(x) = b$ and $\lim_{y\to b} g(y) = c$, and assume that $f(x) \neq b$ for all $x \in A \setminus \{a\}$ in a neighborhood of a. Then $\lim_{x\to a}(g\circ f)(x) = c$.*

An ε-δ proof:

Proof. Let $\varepsilon > 0$. Find $\delta > 0$ such that $|g(y) - c| < \varepsilon$ for all $y \in B \setminus \{b\}$ with $|y - b| < \delta$. Find $\tau > 0$ such that $|f(x) - b| < \delta$ for all $x \in A \setminus \{a\}$ with $|x - a| < \tau$. For such x, from the neighborhood of a where $f(x) \neq b$, since $f(x) \in B \setminus \{b\}$ and $|f(x) - b| < \delta$, we have $|g(f(x)) - c| < \varepsilon$. ■

A sequential proof:

Proof. Let (x_n) be a sequence in $A \setminus \{a\}$ with $x_n \longrightarrow a$. Then $f(x_n)$ is a sequence in B, and $f(x_n) \neq b$ for n large enough (since $f(x) \neq b$ for all $x \in A \setminus \{a\}$ in a neighborhood of a). So, $g(f(x_n)) \longrightarrow c$. ■

3.3. Infinite limits

Infinite limits also make sense for functions: Let $f\colon A \longrightarrow \mathbb{R}$ and let a be a limit point of A; we write $\lim_{x\to a} f(x) = +\infty$ or $f(x) \longrightarrow +\infty$ as $x \longrightarrow a$ if for any M there exists $\delta > 0$ such that $f(x) > M$ for all $x \in A \setminus \{a\}$ with $|x - a| < \delta$; we write $\lim_{x\to a} f(x) = -\infty$ or $f(x) \longrightarrow -\infty$ as $x \longrightarrow a$ if for any M there exists $\delta > 0$ such that $f(x) < M$ for all $x \in A \setminus \{a\}$ with $|x - a| < \delta$; and we write $\lim_{x\to a} f(x) = \infty$ or $f(x) \longrightarrow \infty$ as $x \longrightarrow a$ if $\lim_{x\to a} |f(x)| = +\infty$.

The equivalent sequential definition of infinite limits is, of course, that $\lim_{x\to a} f(x) = +\infty$, $-\infty$, or ∞ if $f(x_n) \longrightarrow +\infty$, $-\infty$, or ∞ respectively for every sequence (x_n) in $A \setminus \{a\}$ with $x_n \longrightarrow a$.

Here is the cimparison principle for infinite limits:

Theorem 3.3.1. *Let $f, g\colon A \longrightarrow \mathbb{R}$ and let a be a limit point of A. If $\lim_{x\to a} f(x) = +\infty$ and $g(x) \geq f(x)$ for all $x \in A \setminus \{a\}$ in a neighborhood of a, then $\lim_{x\to a} g(x) = +\infty$. If $\lim_{x\to a} f(x) = -\infty$ and $g(x) \leq f(x)$ for all $x \in A \setminus \{a\}$ in a neighborhood of a, then $\lim_{x\to a} g(x) = -\infty$. If $\lim_{x\to a} f(x) = \infty$ and $|g(x)| \geq |f(x)|$ for all $x \in A \setminus \{a\}$ in a neighborhood of a, then $\lim_{x\to a} g(x) = \infty$.*

Proof (of the first statement only)**.** If $g(x) \geq f(x)$ for all $x \in A \setminus \{a\}$ in a neighborhood of a and $f(x) \geq M$ for all $x \in A \setminus \{a\}$ in a neighborhood of a, then $g(x) \geq M$ for all $x \in A \setminus \{a\}$ in the smallest of these two neighborhoods. ■

And, here are some theorems on the arithmetic of infinite limits, $1/0 = \infty$, $1/\infty = 0$, $b + \infty = \infty$, $b\infty = \infty$ if $b \neq 0$:

Theorem 3.3.2. *Let $f\colon A \longrightarrow \mathbb{R}$ and let a be a limit point of A. If $\lim_{x\to a} f(x) = 0$ and $f(x) \neq 0$ for all $x \in A\setminus\{a\}$ in a neighborhood of a, then $\lim_{x\to a}(1/f(x)) = \infty$. If $\lim_{x\to a} f(x) = \infty$ then $\lim_{x\to a}(1/f(x)) = 0$.*

Proof. Let $\lim_{x\to a} f(x) = 0$ and $f(x) \neq 0$ in a neighborhood of a. (The second condition guarantees that $1/f$ is defined for all $x \in A \setminus \{a\}$ in a neighborhood of a.) Let $M > 0$. Find $\delta > 0$ such that $|f(x)| < 1/M$ and $f(x) \neq 0$ for all $x \in A\setminus\{a\}$ with $|x-a| < \delta$. Then $|1/f(x)| > M$ for all these x, so $\lim_{x\to a}(1/f(x)) = \infty$.

Now let $\lim_{x\to a} f(x) = \infty$. Let $\varepsilon > 0$. Find $\delta > 0$ such that $|f(x)| > 1/\varepsilon$ for all $x \in A \setminus \{a\}$, then $f(x) \neq 0$ and $|1/f(x)| < \varepsilon$ for all such x. So, $\lim_{x\to a}(1/f(x)) = 0$. ■

Theorem 3.3.3. *Let $f, g\colon A \longrightarrow \mathbb{R}$ and let a be a limit point of A. If $\lim_{x\to a} f(x) = +\infty$ and $\lim_{x\to a} g(x) = b \in \mathbb{R}$, then $\lim_{x\to a}(f(x) + g(x)) = +\infty$, and if $b \neq 0$ then $\lim_{x\to a}(f(x)g(x)) = +\infty$ for $b > 0$ and $\lim_{x\to a}(f(x)g(x)) = -\infty$ for $b < 0$. And if $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = +\infty$, then $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a}(f(x)g(x)) = +\infty$.*

(Of course, a similar theorem holds when $\lim_{x\to a} f(x) = -\infty$ or ∞. It is not however true that if $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a}(f(x) + g(x)) = \infty$ or even exists.)

Proof. Given $M \in \mathbb{R}$, find $\delta > 0$ such that $f(x) > M - (b - 1)$ and $|g(x) - b| < 1$ for all $x \in A \setminus \{a\}$ with $|x - a| < \delta$. Then $g(x) > b - 1$ and so, $f(x) + g(x) > M$ for all such x. So, $\lim_{x\to a}(f(x) + g(x)) = +\infty$.

Let $b > 0$. Given $M \in \mathbb{R}$, find $\delta > 0$ such that $f(x) > 2M/b$ and $g(x) > b - b/2 = b/2$ for all $x \in A \setminus \{a\}$ with $|x - a| < \delta$. Then $f(x)g(x) > (2M/b)(b/2) = M$ for all such x. So, $\lim_{x\to a}(f(x)g(x)) = +\infty$.

Let $b < 0$, then $\lim_{x\to a}(-g(x)) = -b > 0$, so $\lim_{x\to a}(-f(x)g(x)) = \lim_{x\to a}(f(x)(-g(x))) = +\infty$, so $\lim_{x\to a}(f(x)g(x)) = -\infty$.

If both $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = +\infty$, given $M \in \mathbb{R}$, find $\delta > 0$ such that $f(x) > M$ and $g(x) > 1$ for all $x \in A \setminus \{a\}$ with $|x-a| < \delta$, then $f(x)+g(x) > M+1 > M$ and $f(x)g(x) > M \cdot 1 = M$ for such x. ■

3.4. Limits at infinity and one-sided limits

You don't need to read this extremely boring section in details, just get the idea.

The meaning of "f has limit b at a" is that when x "is close" to a, $f(x)$ "is close" to b. "To be close to $+\infty$" means to be larger than "a large" $N \in \mathbb{R}$; we've already used this concept when defined $\lim_{x\to a} f(x) = +\infty$. We may also define the limit of a function, finite or infinite, *at* ∞:

Let $f\colon A \longrightarrow \mathbb{R}$.

(i) Let A be unbounded above; we write $\lim_{x\to+\infty} f(x) = b$ or $f(x) \longrightarrow b$ as $x \longrightarrow +\infty$ if for any $\varepsilon > 0$ there exists N such that $|f(x) - b| < \varepsilon$ for all $x \in A$ such that $x > N$; we write $\lim_{x\to+\infty} f(x) = +\infty$ or $f(x) \longrightarrow +\infty$ as $x \longrightarrow +\infty$ if for any M there exists N such that $f(x) > M$ for all $x \in A$ such that $x > N$; we write $\lim_{x\to+\infty} f(x) = -\infty$ or $f(x) \longrightarrow -\infty$ as $x \longrightarrow +\infty$ if for any M there exists N such that $f(x) < M$ for all $x \in A$ such that $x > N$; and we write $\lim_{x\to+\infty} f(x) = \infty$ or $f(x) \longrightarrow \infty$ as $x \longrightarrow +\infty$ if $\lim_{x\to+\infty} |f(x)| = +\infty$.

(ii) Similarly, let A be unbounded below; we write $\lim_{x\to-\infty} f(x) = b$ or $f(x) \longrightarrow b$ as $x \longrightarrow -\infty$ if for any $\varepsilon > 0$ there exists N such that $|f(x) - b| < \varepsilon$ for all $x \in A$ such that $x < N$; we write $\lim_{x\to-\infty} f(x) = +\infty$ or $f(x) \longrightarrow +\infty$ as $x \longrightarrow -\infty$ if for any M there exists N such that $f(x) > M$ for all $x \in A$ such that $x < N$; we write $\lim_{x\to-\infty} f(x) = -\infty$ or $f(x) \longrightarrow -\infty$ as $x \longrightarrow -\infty$ if for any M there exists N such that $f(x) < M$ for all $x \in A$ such that $x < N$; and we write $\lim_{x\to-\infty} f(x) = \infty$ or $f(x) \longrightarrow \infty$ as $x \longrightarrow -\infty$ if $\lim_{x\to-\infty} |f(x)| = +\infty$.

(iii) Finally, let A be unbounded; we write $\lim_{x\to\infty} f(x) = b$ or $f(x) \longrightarrow b$ as $x \longrightarrow \infty$ if for any $\varepsilon > 0$ there exists N such that $|f(x) - b| < \varepsilon$ for all $x \in A$ such that $|x| > N$; we write $\lim_{x\to\infty} f(x) = +\infty$ or $f(x) \longrightarrow +\infty$ as $x \longrightarrow \infty$ if for any M there exists N such that $f(x) > M$ for all $x \in A$ such that $|x| > N$; we write $\lim_{x\to\infty} f(x) = -\infty$ or $f(x) \longrightarrow -\infty$ as $x \longrightarrow \infty$ if for any M there exists N such that $f(x) < M$ for all $x \in A$ such that $|x| > N$; and we write $\lim_{x\to\infty} f(x) = \infty$ or $f(x) \longrightarrow \infty$ as $x \longrightarrow \infty$ if $\lim_{x\to\infty} |f(x)| = +\infty$. (Ufff...)

Another variant of limits at a point is *the left- and the right-sided limits*, which also may be finite and infinite:

Let $f\colon A \longrightarrow \mathbb{R}$.

(iv) Let a be a limit point of $A \cap \{x : x < a\}$. We say that *the left-sided*, or *the left-hand* limit of f at a is b and write $\lim_{x\to a^-} f(x) = b$ or $f(x) \longrightarrow b$ as $x \longrightarrow a^-$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - b| < \varepsilon$ for all $x \in A$ such that $a - \delta < x < a$; we write $\lim_{x\to a^-} f(x) = +\infty$ or $f(x) \longrightarrow +\infty$ as $x \longrightarrow a^-$ if for any M there exists $\delta > 0$ such that $f(x) > M$ for all $x \in A$ such that $a - \delta < x < a$; we write $\lim_{x\to a^-} f(x) = -\infty$ or $f(x) \longrightarrow -\infty$ as $x \longrightarrow a^-$ if for any M there exists $\delta > 0$ such that $f(x) < M$ for all $x \in A$ such that $a - \delta < x < a$; and we write $\lim_{x\to a^-} f(x) = \infty$ or $f(x) \longrightarrow \infty$ as $x \longrightarrow a^-$ if $\lim_{x\to a^-} |f(x)| = +\infty$.

(v) Now let a be a limit point of $A \cap \{x : x > a\}$. We say that *the right-sided*, or *the right-hand* limit of f at a is b and write $\lim_{x\to a^+} f(x) = b$ or $f(x) \longrightarrow b$ as $x \longrightarrow a^+$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - b| < \varepsilon$ for all $x \in A$ such that $a < x < a + \delta$; we write $\lim_{x\to a^+} f(x) = +\infty$ or $f(x) \longrightarrow +\infty$ as $x \longrightarrow a^+$ if for any M there exists $\delta > 0$ such that $f(x) > M$ for all $x \in A$ such that $a < x < a + \delta$; we write $\lim_{x\to a^+} f(x) = -\infty$ or $f(x) \longrightarrow -\infty$ as $x \longrightarrow a^+$ if for any M there exists $\delta > 0$ such that $f(x) < M$ for all $x \in A$ such that $a < x < a + \delta$; and we write $\lim_{x\to a^+} f(x) = \infty$ or $f(x) \longrightarrow \infty$ as $x \longrightarrow a^+$ if $\lim_{x\to a^+} |f(x)| = +\infty$.

Clearly, f has a limit at a point iff it has both one-sided limits (if they make sense) and these limits coincide:

Theorem 3.4.1. *Let $f\colon A \longrightarrow \mathbb{R}$ and let a be a limit point of both $A \cap \{x : x < a\}$ and $A \cap \{x : x > a\}$. Then $\lim_{x\to a} f(x)$ exists iff both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and are equal.*

Proof. I'll only prove the case of a finite limit. Let $\lim_{x\to a} f(x) = b$. Given $\varepsilon > 0$, find $\delta > 0$ such that $|f(x) - b| < \varepsilon$ for all $x \in A \setminus \{a\}$ with $|x - a| < \delta$. Then we have $|f(x) - b| < \varepsilon$ for all $x \in A$ such that $a - \varepsilon < x < a$, and for all $x \in A$ such that $a < x < a + \varepsilon$. Hence, $\lim_{x\to a^-} f(x) = b$ and $\lim_{x\to a^+} f(x) = b$.

Conversely, assume that $\lim_{x\to a^-} f(x) = b$ and $\lim_{x\to a^+} f(x) = b$. Find $\delta_1 > 0$ such that $|f(x) - b| < \varepsilon$ for all $x \in A$ with $a - \delta_1 < x < a$. and find $\delta_2 > 0$ such that $|f(x) - b| < \varepsilon$ for all $x \in A$ with $a < x < a + \delta_2$. Put $\delta = \min\{\delta_1, \delta_2\}$; then $|f(x) - b| < \varepsilon$ for all $x \in A \setminus \{a\}$ with $|x - a| < \delta$. So, $\lim_{x\to a} f(x) = b$. ■

Of course, the one-sided limits, as well as the limits at infinity, can be introduced via sequences. Let's write $x_n \longrightarrow a^-$ if $x_n \longrightarrow a$ and $x_n < a$ for all n and $x_n \longrightarrow a^+$ if $x_n \longrightarrow a$ and $x_n > a$ for all n. Let α and β be any of a (where $a \in \mathbb{R}$), a^-, a^+, $+\infty$, $-\infty$, and ∞. Then $\lim_{x\to\alpha} f(x) = \beta$ iff for any sequence (x_n) in $\mathrm{Dom}(f) \setminus \{a\}$ with $x_n \longrightarrow \alpha$ we have $f(x_n) \longrightarrow \beta$.

For the limits of the form $\lim_{x\to+\infty}$, $\lim_{x\to-\infty}$, $\lim_{x\to\infty}$, $\lim_{x\to a^-}$, and $\lim_{x\to a^+}$, all Theorems 3.2.1 – 3.3.3 remain true, but, sometimes, need some adaptation. We cannot, for example, claim that if a finite $\lim_{x\to a^-} f(x)$ exists, then f is bounded in a neighborhood of a, but only that it is bounded in "a left neighborhood" of a, that is, in an interval $(a - \delta, a)$.

Since $\lim_{x\to+\infty} x = +\infty$, we have by induction on n that $\lim_{x\to+\infty} x^n = +\infty$ for any $n \in \mathbb{N}$; since for any even n, $(-x)^n = x^n$, $\lim_{x\to-\infty} x^n = +\infty$; since for any odd n, $(-x)^n = -x^n$, $\lim_{x\to-\infty} x^n = -\infty$. It follows that for any $n \in \mathbb{N}$, $\lim_{x\to 0^+} x^{-n} = \lim_{x\to 0^+} \frac{1}{x^n} = +\infty$; if n is even, then $\lim_{x\to 0^-} x^{-n} = +\infty$; if n is odd, $\lim_{x\to 0^-} x^{-n} = -\infty$; and for any n, $\lim_{x\to\infty} x^{-n} = 0$.

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial, where $a_n, \ldots, a_0 \in \mathbb{R}$ and $a_n \neq 0$. Then $f(x) = x^n(a_n + a_{n-1}x^{-1} + \cdots + a_1 x^{-n+1} + a_0 x^{-n})$; since $\lim_{x\to\infty}(a_n + a_{n-1}x^{-1} + \cdots + a_1 x^{-n+1} + a_0 x^{-n}) = a_n > 0$, we obtain that: if n is odd and $a_n > 0$, then $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to+\infty} f(x) = +\infty$; if n is odd and $a_n < 0$, then $\lim_{x\to-\infty} f(x) = +\infty$ and $\lim_{x\to+\infty} f(x) = -\infty$; if n is even and $a_n > 0$, then $\lim_{x\to-\infty} f(x) = +\infty$ and $\lim_{x\to+\infty} f(x) = +\infty$; and if n is even and $a_n < 0$, then $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to+\infty} f(x) = -\infty$.

3.5. Monotone functions

We say that a function $f\colon A \longrightarrow \mathbb{R}$ (where $A \subseteq \mathbb{R}$) is *increasing* if for any $x, y \in A$ with $x < y$ we have $f(x) \leq f(y)$; *strictly increasing* if for any $x, y \in A$ with $x < y$ we have $f(x) < f(y)$; *decreasing* if for any $x, y \in A$ with $x < y$ we have $f(x) \geq f(y)$; and *strictly decreasing* if for any $x, y \in A$ with $x < y$ we have $f(x) > f(y)$. (In the book, – *nondecreasing*, *increasing*, *nonincreasing*, and *decreasing* respectively.) A function is said to be *monotone* if it is increasing or is decreasing, and *strictly monotone* if it is strictly increasing of strictly decreasing.

A nice fact about monotone functions is that they have one-sided limits at every limit point of their domain:

Theorem 3.5.1. *Let $f\colon A \longrightarrow \mathbb{R}$ be a monotone function, let $a \in \mathbb{R}$. If a is a limit point of $A \cap \{x : x < a\}$ then $\lim_{x\to a^-} f(x)$ exists (finite of infinite), and if $a \in A$ then $\lim_{x\to a^-} f(x) \leq f(a)$ if f is increasing and $\lim_{x\to a^-} f(x) \geq f(a)$ if f is decreasing. If a is a limit point of $A \cap \{x : x > a\}$ then $\lim_{x\to a^+} f(x)$ exists (finite of infinite), and if $a \in A$ then $\lim_{x\to a^+} f(x) \geq f(a)$ if f is increasing and $\lim_{x\to a^+} f(x) \leq f(a)$ if f is decreasing. Also, if A is unbounded above, then $\lim_{x\to+\infty} f(x)$ exists (finite of infinite); if A is unbounded below, then $\lim_{x\to-\infty} f(x)$ exists.*

Proof. W.l.o.g., let's assume that f is increasing. Let a be a limit point of $A \cap \{x : x < a\}$. Put $b = \sup\{f(x) \mid x < a\}$. Assume first that b is finite, $b \in \mathbb{R}$. Given $\varepsilon > 0$, find $x_0 < a$ such that $f(x_0) > b - \varepsilon$. Put $\delta = a - x_0$. Then for any $x \in A$ with $x_0 = a - \delta < x < a$ we have $b - \varepsilon < f(x_0) \leq f(x) \leq b$. So, $\lim_{x\to a^-} f(x) = b$. Notice also that if $a \in A$, then $f(x) \leq f(a)$ for all $x \in A$ with $x < a$, so $b \leq f(a)$.

Now assume that $b = +\infty$. Given $M \in \mathbb{R}$, find $x_0 < a$ such that $f(x_0) > M$. Put $\delta = a - x_0$. Then for any $x \in A$ with $x_0 = a - \delta < x < a$ we have $M < f(x_0) \leq f(x)$. So, $\lim_{x\to a^-} f(x) = +\infty$.

All other assertions can be proved similarly. ■

The Cantor ladder function $f\colon [0,1] \longrightarrow [0,1]$ is constructed in the following way: We put $f(0) = 0$ and $f(1) = 1$. Then we put $f(x) = 1/2$ for all $x \in \left[\frac{1}{3}, \frac{2}{3}\right]$, then $f(x) = 1/4$ for all $x \in \left[\frac{1}{9}, \frac{2}{9}\right]$ and $f(x) = 3/4$ for all $x \in \left[\frac{7}{9}, \frac{8}{9}\right]$, etc. (In the process of constructing the Cantor set C, we define f to be $\frac{e_1}{2} + \cdots + \frac{e_n}{2^n} + \frac{1}{2^{n+1}}$ on each "removed" interval $\left[\frac{2e_1}{3} + \cdots + \frac{2e_n}{3^n} + \frac{1}{3^{n+1}}, \frac{2e_1}{3} + \cdots + \frac{2e_n}{3^n} + \frac{2}{3^{n+1}}\right]$, $e_i \in \{0,1\}$.) This way f is defined at all points of $[0,1] \setminus C$ (as well as at the boundary points of C). We then extend f to C by putting $f(x) = \sup\bigl\{f(z) : z < x,\ z \in [0,1] \setminus C\bigr\}$ for every $x \in C$. It is clear that the obtained function f is increasing and takes all values of the form $\frac{m}{2^n} \in [0,1]$, $m, n \in \mathbb{N}$. We will see later that f actually takes all values in

$[0,1]$, and, moreover, $f(C) = [0,1]$.

3.6. A Cauchy criterion for functional limits

We can also formulate the Cauchy criterion for existence of a finite limit. (I'll only do it for the case of a limit at a point, but, of course, a similar criterion can be stated for limits at infinity.)

Theorem 3.6.1. *Let $f\colon A \longrightarrow \mathbb{R}$ be a function and let a be a limit point of A. The finite limit $\lim_{x\to a} f(x)$ exists iff for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in A \setminus \{a\}$ with $|x-a|, |y-a| < \delta$ one has $|f(x) - f(y)| < \varepsilon$.*

Proof. Assume that $b = \lim_{x\to a} f(x) \in \mathbb{R}$ exists. Let $\varepsilon > 0$; find $\delta > 0$ such that $|f(x) - b| < \varepsilon/2$ whenever $x \in A \setminus \{a\}$ is such that $|x - a| < \delta$. Then for any $x, y \in A \setminus \{a\}$ such that $|x-a|, |y-a| < \delta$ we have $|f(x) - f(y)| \leq |f(x) - b| + |f(y) - b| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Conversely, assume that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in A \setminus \{a\}$ with $|x-a|, |y-a| < \delta$ one has $|f(x) - f(y)| < \varepsilon$. Let's construct a sequence of closed bounded intervals in $\mathbb{R}$: Find $\delta_1 > 0$ such that for any $x, y \in A \setminus \{a\}$ with $|x-a|, |y-a| < \delta_1$ we have $|f(x) - f(y)| < 1$, pick any point $x_1 \in A \setminus \{a\}$ with $|x_1 - a| < \delta_1$ and put $I_1 = \big[f(x_1) - 1, f(x_1) + 1\big]$, then for any $x \in A \setminus \{a\}$ with $|x - a| < \delta_1$ we have $f(x) \in I_1$. Then by induction, for every $n \in \mathbb{N}$ find $\delta_n > 0$ with $\delta_n \leq \delta_{n-1}$ such that for any $x, y \in A \setminus \{a\}$ with $|x-a|, |y-a| < \delta_n$ we have $|f(x) - f(y)| < 1/n$, pick any point $x_n \in A \setminus \{a\}$ with $|x_n - a| < \delta_n$ and put $I_n = \big[f(x_n) - 1/n, f(x_n) + 1/n\big] \cap I_{n-1}$; then for any $x \in A \setminus \{a\}$ with $|x - a| < \delta_n$ we have $|f(x) - f(x_n| < 1/n$ and, since $\delta_n \leq \delta_{n-1}$, also $f(x) \in I_{n-1}$, so $f(x) \in I_n$.

Now, $I_1 \supseteq I_2 \supseteq \cdots$ is a nested sequence of closed bounded intervals, with $|I_n| \leq 2/n$, thus there is $b \in \bigcap_{n=1}^{\infty} I_n$, and I claim that $b = \lim_{x\to a} f(x)$. Indeed, let $\varepsilon > 0$, let $n \in \mathbb{N}$ be such that $2/n < \varepsilon$. Then for any $x \in A \setminus \{a\}$ with $|x - a| < \delta_n$ we have $f(x) \in I_n$ and $b \in I_n$, so $|f(x) - b| \leq |I_n| \leq 2/n < \varepsilon$. ■

Another proof of the second part. Assume that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in A \setminus \{a\}$ with $|x-a|, |y-a| < \delta$ one has $|f(x) - f(y)| < \varepsilon$. Then for any sequence (x_n) in $A \setminus \{a\}$ converging to a the sequence $(f(x_n))$ is Cauchy and, therefore, converges. If for two such sequences, (x_n) and (y_n), the sequences $(f(x_n))$ and $(f(y_n))$ converge to distinct limits, then for the sequence $(x_1, y_1, x_2, y_2, \ldots)$, which also converges to a, the sequence $\big(f(x_1), f(y_1), f(x_2), f(y_2), \ldots\big)$ diverges. This cannot be, so for all sequences (x_n) in $A \setminus \{a\}$ with $\lim x_n = a$ the corresponding sequences $(f(x_n))$ converge to the same limit b. Hence, $\lim_{x\to a} f(x) = b$. ■

4. Continuous functions

4.1. Functions continuous at a point

Let $f\colon A \longrightarrow \mathbb{R}$ (where $A \subseteq \mathbb{R}$) and let $a \in A$. We say that f *is continuous* at a if a is an isolated point of A, or if a is a limit point of A and $\lim_{x\to a} f(x) = f(a)$. By this definition, f is continuous at a iff for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in A$ with $|x - a| < \delta$, $|f(x) - f(a)| < \varepsilon$; and iff for any sequence (x_n) in A with $x_n \longrightarrow a$, $f(x_n) \longrightarrow f(a)$.

Also, a function f is said to be *left-continuous* at a if $\lim_{x\to a^-} f(x) = f(a)$ and *right-continuous* at a if $\lim_{x\to a^+} f(x) = f(a)$. Clearly, f is continuous at a iff it is both left and right continuous at a.

Here are some properties of functions continuous at a point, which follow directly from Theorems 3.2.3, 3.2.4 and 3.2.6.

Theorem 4.1.1. *If a function f is continuous at a point a, it is bounded in a neighborhood of a.*

Theorem 4.1.2. *Let a function f be continuous at a point a. If $f(a) < b$ for some $b \in \mathbb{R}$, then $f(x) < b$ for all x in a neighborhood of a, and if $f(a) > b$ for some $b \in \mathbb{R}$, then $f(x) > b$ for all x in a neighborhood of a.*

Theorem 4.1.3. *Let functions f and g, with common domain, be continuous at a point a. Then the functions $f + g$, fg, and g/f if $f(a) \neq 0$, are also continuous at a.*

(Notice that if $f(a) \neq 0$, then $f(x) \neq 0$ in a neighborhood of a, then the function g/f is defined in this neighborhood.)

Theorem 4.1.4. *Let $f\colon A \longrightarrow \mathbb{R}$ and $g\colon B \longrightarrow \mathbb{R}$ be functions with $f(A) \subseteq B$, let a be a limit point of A, let $\lim_{x\to a} f(x) = b \in B$, and let g be continuous at b. Then $\lim_{x\to a}(g\circ f)(x) = g(b)$.*

This theorem is pretty close to Theorem 3.2.8, but I'll reprove it anyway:

Proof. Let (x_n) be a sequence in $A \setminus \{a\}$ with $x_n \longrightarrow a$, then $\lim f(x_n) = b$, then $\lim g(f(x_n)) = g(b)$. ■

As a corollary, we get:

Theorem 4.1.5. *If a function f is continuous at a point a and a function g, with $\mathrm{Rng}(f) \subseteq \mathrm{Dom}(g)$, is continuous at $f(a)$, then $g\circ f$ is continuous at a.*

Proof. We have $\lim_{x\to a} f(x) = f(a)$, so $\lim_{x\to a} g(f(x)) = g(f(a))$. ■

We say that f *is continuous on a subset* C of $\mathrm{Dom}(f)$ if f is continuous at every point $a \in C$. We say that f *is continuous* if f is continuous on $\mathrm{Dom}(f)$.

From Theorems 4.1.3 and 4.1.5 we immediately obtain:

Theorem 4.1.6. *If f and g are continuous functions on a subset $C \subseteq \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$, then $f+g$ and fg are also continuos on C, and g/f is continuous on $C \setminus \{x : f(x) \neq 0\}$.*

Theorem 4.1.7. *If function f is continuous on a set C and function g, with $\mathrm{Rng}(f) \subseteq \mathrm{Dom}(g)$, is continuous on $f(C)$, then $g\circ f$ is continuous on C.*

Any constant function $f(x) = c \in \mathbb{R}$, $x \in \mathbb{R}$, and the identity function $f(x) = x$, $x \in \mathbb{R}$, are continuous. By the theorems above, and by induction, the function $f(x) = x^n$ is continuous for any $n \in \mathbb{N}$, any polynomial $f(x) = c_n x^n + \cdots + c_1 x + c_0$ is continuous, any rational function $h(x) = f(x)/g(x)$, where g and h are polynomials, is continuous.

We proved that if $x_n \longrightarrow a$ and $x_n \geq 0$ for all n, then $\sqrt{x_n} \longrightarrow \sqrt{a}$; this shows that $f(x) = \sqrt{x}$, $x \geq 0$, is continuous.

By the theorems above, the function $\sqrt{x^2+1} + 1/\sqrt{3x^5 + 10/x}$ is also continuous.

4.2. Discontinuous functions

If f isn't continuous at $a \in A$ we say that f *is discontinuous* at a. If $\lim_{x\to a} f(x)$ exists but $\neq f(a)$, we say that the discontinuity of f at a is *removable*. If both $b_1 = \lim_{x\to a^-} f(x)$ and $b_2 = \lim_{x\to a^+} f(x)$ exist and are finite, but $b_1 \neq b_2$, we say that the discontinuity of f at a is *of the first kind*, or *a jump discontinuity*. If f is discontinuous at a but its discontinuity is neither removable nor of the first kind, we say that f has discontinuity *of the second kind* at f.

The function $f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$ is continuous at all points except 0, and has a removable discontinuity at 0.

The function $\mathrm{sign}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$ is continuous at all points except 0, and has a discontinuity of the first kind at 0.

The function $f(x) = \begin{cases} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ (the function sin will be defined later) is continuous at all points except 0, and has a discontinuity of the second kind at 0.

Here are two interesting functions:

Dirichlet's function is defined by $f(x) = \begin{cases} 0, & x \in \mathbb{R} \setminus \mathbb{Q} \\ 1, & x \in \mathbb{Q} \end{cases}$; this function is discontinuous at all points: indeed, for any $a \in \mathbb{R}$, any neighborhood of a contains rational points x, where $f(x) = 1$, and irrational points x, where $f(x) = 0$.

Riemann's function is defined by $f(x) = \begin{cases} 0, & x \in \mathbb{R} \setminus \mathbb{Q} \\ 1/m, & x \in \mathbb{Q},\ x = n/m \text{ in the lowest terms} \end{cases}$; this function is continuous at all irrational points and discontinuous at all rational points. Indeed, if $a \in \mathbb{Q}$, then $f(a) > 0$, but any neighborhood of a contains points x where $f(x) = 0$; hence, f is discontinuous at a. If $a \notin \mathbb{Q}$, then $f(a) = 0$. Let $\varepsilon > 0$; find $k \in \mathbb{N}$ such that $1/k < \varepsilon$, put $S = \{n/m : n \in \mathbb{Z},\ m \in \mathbb{N},\ m < k\}$. Let I be the interval $[a-2, a+2]$. Then $S \cap I$ is a finite set; let $\delta = \min\bigl\{|a-r| : r \in S\bigr\}$, then $\delta > 0$. Then for any $x \in (a-\delta, a+\delta)$, either $f(x) = 0$, or $f(x) = 1/m$ with $m \geq k$, so $0 < f(x) \leq 1/k < \varepsilon$. So, $\lim_{x \to a} f(x) = 0 = f(a)$.

It is interesting that there is no function which is, on the contrary, continuous at all rational points and discontinuous at all irrational points. Indeed, let f be a function that is continuous on $\mathbb{Q}$. For every $a \in \mathbb{R}$ define *the variation of f at a*, $\mathrm{Var}_a f = \lim_{\delta \to 0^+} \sup\bigl\{|f(x) - f(y)| : |x-a|, |y-a| < \delta\bigr\}$. It is easy to see that $\mathrm{Var}_a f = 0$ iff f is continuous at a, so, in our case, $\mathrm{Var}_a f = 0$ for all $a \in \mathbb{Q}$. Also, it is easy to see that if $\mathrm{Var}_a f = 0$ for some a, then for any $\varepsilon > 0$ there is $\delta > 0$ such that $\mathrm{Var}_x f < \varepsilon$ for all x with $|x-a| < \delta$. (This says that the function $\varphi(x) = \mathrm{Var}_x f$ is *upper semicontinuous*.) Since every interval contains a rational point, it contains, for any $\varepsilon > 0$, a subinterval where $\mathrm{Var} f < \varepsilon$. (This says that the set $\{a : \mathrm{Var}_a f \geq \varepsilon\}$ is *nowehere dense*.) Now, for every $n \in \mathbb{N}$, let $A_n = \bigl\{a : \mathrm{Var}_a f \geq 1/n\bigr\}$. Then $\mathbb{R} = \bigl(\bigcup_{n=1}^{\infty} A_n\bigr) \cup \mathbb{Q}$. *The Baire category theorem* says that this is impossible: $\mathbb{R}$ is not representable as a countable union of nowhere dense sets. Indeed, count $\mathbb{Q}$, $\mathbb{Q} = \{r_1, r_2, \ldots\}$. Find a closed interval I_1 such that $I_1 \cap A_1 = \emptyset$ and $r_1 \notin I_1$. Find a closed subinterval I_2 of I_1 such that $I_2 \cap A_2 = \emptyset$ and $r_2 \notin I_2$. Continue by induction, and let $a \in \bigcap_{n=1}^{\infty} I_n$. Then $a \neq r_n$ for all n, so a is irrational. However, also $a \notin A_n$ for all n, so $\mathrm{Var}_a f = 0$. Hence, f is continuous at an irrational point.

Theorem 3.5.1 says that any monotone function f may only have discontinuities of the first kind. Every point a of discontinuity of f defines "a gap" in $\mathrm{Rng}(f)$: the interval $\bigl(\lim_{x \to a^-} f(x), \lim_{x \to a^+} f(x)\bigr)$ contains at most one point of $\mathrm{Rng}(f)$, $f(a)$. Thus if the range of a monotone function has no such "gaps", the function is continuous:

Theorem 4.2.1. *Let f be a monotone function such that* $\mathrm{Rng}(f)$ *is dense in the interval* $\bigl(\inf \mathrm{Rng}(f), \sup \mathrm{Rng}(f)\bigr)$*; then f is continuous.*

It follows that the Cantor ladder function (constructed in the end of subsection 3.5) is continuous: it is increasing and takes all binary rational values (numbers of the form $\frac{m}{2^n}$) in $[0,1]$, which are dense in $[0,1]$.

If a function f is strictly increasing or decreasing, then f is injective and the inverse function $f^{-1}\colon \mathrm{Rng}(f) \longrightarrow \mathbb{R}$ is also strictly increasing: Indeed, if f is strictly increasing, $u = f(x)$, $v = f(y)$ and $u < v$, then $x = f^{-1}(u) < y = f^{-1}(v)$ (since $x \geq y$ would imply $f(x) \geq f(y)$). Thus we easily obtain the following fact:

Theorem 4.2.2. *Let f be a strictly monotone function on an interval I; then the inverse function f^{-1} is continuous on $f(I)$.*

Proof. f^{-1} is strictly monotone and $\mathrm{Rng}(f^{-1})$ is the interval I. ■

If f is a monotone function, each its point of discontinuity defines "a gap" in $\mathrm{Rng}(f)$ and these "gaps" are disjoint intervals; so, the set of gaps, and therefore, the set of points of discontinuities of f is at most countable. We obtain the following nice result:

Theorem 4.2.3. *A monotone function may only have at most countably many discontinuities.*

4.3. The intermediate value theorem

The intermediate value theorem, *the I.V.T.*, says that if a function continuous on an interval takes two values, it also takes all values between these two:

Theorem 4.3.1. (IVT) *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a continuous function. If $c \in \mathbb{R}$ is such that $f(a) < c < f(b)$ or $f(b) < c < f(a)$, then $c = f(x_0)$ for some $x_0 \in (a,b)$.*

Proof. W.l.o.g., assume that $f(a) < f(b)$, and let $f(a) < c < f(b)$. Define $A = \{x \in I : f(x) < c\}$. Then $A \subseteq I$, $a \in A$ and $b \notin A$. A is bounded above (by b); let $x_0 = \sup A$. I claim that $f(x_0) = c$. Indeed, since $x_0 = \sup A$, every neighborhood of x_0 contains points of A; choose a sequence (x_n) in A with $x_n \longrightarrow x_0$, then $\lim f(x_n) \longrightarrow f(x_0)$ and $f(x_n) < c$ for all n, so $f(x_0) \leq c$. On the other hand, for any $x \in I$ with

$x > x_0$, $f(x) > c$, so $f(x_0) \geq c$ by continuity. (If $x_0 = b$, then there is no such x, but then $f(x_0) > c$.) Hence, $f(x_0) = c$. ■

The I.V.T. can be reformulated as follows:

Theorem 4.3.2. *If I is an interval (bounded or unbounded) and a non-constant function $f: I \longrightarrow \mathbb{R}$ is continuous, then $J = \mathrm{Rng}(f) = f(I)$ is also an interval.*

Proof. Let $m = \inf f(I)$ and $M = \sup f(I)$ (which may be infinite); since f is non-constant, $m \neq M$. If $m, M \in J$, there are $a, b \in I$ such that $f(a) = m$ and $f(b) = M$, and by the I.V.T. for every $y \in (m, M)$ there is $x \in (a, b)$ (or (b, a)) such that $f(x) = y$; so, $J = [m, M]$.

If, say, $m \in J$ but $M \notin J$, then there is $a \in I$ such that $f(a) = m$. Let $m < y < M$; find $b \in I$ such that $y < f(b) < M$; then by the I.V.T., there is $x \in (a, b)$ (or (b, a)) such that $f(x) = y$. Hence, $J = [m, M)$ in this case.

The cases $m \notin J$, $M \in J$ and $m \notin J$, $M \notin J$ are similar. ■

For any $n \in \mathbb{N}$ and $a \in \mathbb{R}$, we say that b *is root of degree n of a* and write $b = \sqrt[n]{a}$ if $b^n = a$, and when n is even, also $b \geq 0$. If $b, c \geq 0$, then $b < c$ iff $b^n < c^n$ (which can be proved by induction), so, for $a \geq 0$, there can be at most one $\sqrt[n]{a}$. We can now prove that for any $n \in \mathbb{N}$ and $a \geq 0$, $\sqrt[n]{a}$ exists:

Theorem 4.3.3. *For any $n \in \mathbb{N}$ and every $a \geq 0$ there exists $b > 0$ such that $b^n = a$.*

Proof. For $a = 0$ we put $b = 0$, so let $a > 0$. The function $f(x) = x^n$ is continuous (by induction on n, as the product of n copies of constinuous function $h(x) = x$). We have $f(0) = 0$, and since $f(x) \longrightarrow +\infty$ as $x \longrightarrow +\infty$, there exists $c > 0$ such that $f(c) > a$. By the I.V.T., there exists $b \in (0, c)$ such that $b^n = f(b) = a$. ■

If n is odd then $\sqrt[n]{a}$ exists for negative a as well, which is just $-\sqrt[n]{-a}$.

As another application of the I.V.T. we can prove that any polynomial of odd degree has a root:

Theorem 4.3.4. *If $n \in \mathbb{N}$ is odd and f is a polynomial of degree n, there exists $c \in \mathbb{R}$ such that $f(c) = 0$.*

Proof. Since $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to +\infty} = +\infty$, or vice versa, there is $a \in \mathbb{R}$ such that $f(a) < 0$ and there is $b \in \mathbb{R}$ such that $f(b) > 0$. Since f is continuous, there exists $c \in [a, b]$ or $[b, a]$ such that $f(c) = 0$. ■

We also see that the Cantor ladder function (which is increasing and continuous on $[0, 1]$) takes all values in $[0, 1]$, and since it is constant on intervals from $[0, 1] \setminus C$, maps the Cantor set C onto $[0, 1]$.

Next, we can obtain the following nice fact:

Theorem 4.3.5. *Any continuous injective function on an interval is strictly monotone.*

Proof. Let I be an interval (bounded or unbounded), let $f: I \longrightarrow \mathbb{R}$ be continuous. Let $a, b, c \in I$, $a < b < c$; I claim that either $f(a) < f(b) < f(c)$ or $f(a) > f(b) > f(c)$, that is, f is strictly monotone on every 3-element subset of I. Indeed, assume that $f(a) < f(b)$. If $f(c) = f(b)$, f is not injective. If $f(c) < f(b)$, choose any y such that $f(a), f(c) < y < f(b)$; then by the I.V.T. there are $u \in (a, b)$ and $v \in (b, c)$ such that $f(u) = y$ and $f(v) = y$, which contradicts the injectivity of f. Hence, $f(b) < f(c)$. The case $f(a) > f(b)$ is similar.

This implies that f is strictly monotone on every 4-element subset of I. Let $a, b, c, d \in I$, $a < b < c < d$. If $f(a) < f(b)$, then $f(a) < f(b) < f(c)$, and then $f(b) < f(c) < f(d)$; if $f(a) > f(b)$, then $f(a) > f(b) > f(c)$, and then $f(b) > f(c) > f(d)$.

Now, choose any $a, b \in I$ with $a < b$. Suppose that $f(a) < f(b)$. For any $x, y \in I$ with $x < y$, f is strictly monotone on the set $\{a, b, x, y\}$; since $f(a) < f(b)$, f is increasing on this set, so $f(x) < f(y)$. Hence, f is strictly increasing on I. Similarly, if $f(a) > f(b)$, then for any $x, y \in I$ with $x < y$ we have $f(x) > f(y)$, so f is strictly decreasing on I. ■

Theorems 4.2.2 and 4.3.5 together imply:

Theorem 4.3.6. *If I is an interval in $\mathbb{R}$ and $f: I \longrightarrow J \subseteq \mathbb{R}$ is continuous and invertible, then $f^{-1}: J \longrightarrow I$ is also continuous.*

Example. For any even $n \in \mathbb{N}$, the function $\sqrt[n]{x}$, which is the inverse of $f(x) = x^n$ as a function $[0, +\infty) \longrightarrow [0, +\infty)$, is continuous. (As a function $\mathbb{R} \longrightarrow [0, +\infty)$, f is not injective and so, not invertible.) For any odd $n \in \mathbb{N}$, the function $\sqrt[n]{x}$, which is the inverse of $f(x) = x^n$ as a function $\mathbb{R} \longrightarrow \mathbb{R}$, is continuous.

4.4. Maximum and minimum of a function continuous on a closed bounded interval

This is an important fact, that holds for any *compact* space (where the Bolzano-Weierstrass theorem, Theorem 2.6.1, takes place):

Theorem 4.4.1. *Let f be a continuous function on a closed bounded interval I. Then f is bounded on I, and attains its maximal and minimal values: there are $p, q \in I$ such that $f(q) \leq f(x) \leq f(p)$ for all $x \in I$.*

Proof. Let $I = [a, b]$. Assume that f is unbounded, say, above. For every $n \in \mathbb{N}$ choose a point $x_n \in [a, b]$ such that $f(x_n) \geq n$. By the Bolzano-Weierstrass theorem, there is a converging subsequence (x_{n_i}) of (x_n); let $x_{n_i} \longrightarrow z$, then $a \leq z \leq b$, that is, $z \in [a, b]$. Since f is continuous at z, we have $f(x_{n_i}) \longrightarrow f(z)$, but $f(x_{n_i}) \geq n_i \longrightarrow +\infty$ as $i \longrightarrow \infty$, contradiction.

So, f is bounded above; let $M = \sup f(I) = \sup\{f(x), x \in I\}$. For every $n \in \mathbb{N}$, find $x_n \in [a, b]$ such that $f(x_n) > M - 1/n$, let (x_{n_i}) be a converging subsequence of (x_n); put $p = \lim_{i \to \infty} x_{n_i}$, then $a \leq p \leq b$, sp $p \in I$. Then $f(p) = \lim_{i \to \infty} f(x_{n_i})$, but since $M - 1/n_i < f(x_{n_i}) \leq M$ for all i, $\lim f(x_{n_i}) = M$, so $f(p) = M$.

The existence of the point of minimum of f can be proved similarly, or follows from the existence of maximum of the function $-f$. ■

Here is another proof (actually, the same proof, but with the two statements being proved simultaneously):

Proof. Let $M = \sup f(I)$ and assume that $M \notin f(I)$ (or, without this assumption, just notice that $f(I)$ is an interval), then there is a sequence (y_n) in $f(I)$ such that $y_n \longrightarrow M$. For each n let $x_n \in I$ be such that $f(x_n) = y_n$. By the Bolzano-Weierstrass theorem, (x_n) has a comverging subsequence (x_{n_i}); let $p = \lim x_{n_i}$, then $p \in I$. (If $I = [a, b]$, since $a \leq x_{n_i} \leq b$ for all i, we have $a \leq p \leq b$ as well.) Since f is continuous at p, $\lim f(x_{n_i}) = f(p)$. But also $\lim f(x_{n_i}) = \lim f(x_n) = \lim y_n = M$, so $f(p) = M$. ■

Combining the I.V.T. and Theorem 4.4.1, we get:

Theorem 4.4.2. *If I is a closed bounded interval and a non-constant function $f \colon I \longrightarrow \mathbb{R}$ is continuous, then* $\mathrm{Rng}(f) = f(I)$ *is also a closed bounded interval.*

Under some additional conditions, Theorem 4.4.1 sometimes applies in a situation where $\mathrm{Dom}(f)$ is not closed or is not bounded:

Theorem 4.4.3. *Let $I = (\alpha, \beta)$ be an interval, bounded or unbounded, let $f \colon I \longrightarrow \mathbb{R}$ be a continuous function with* $\lim_{x \to \alpha^+} f(x) = \lim_{x \to \beta^-} f(x) = +\infty$. *Then there exists $q \in I$ such that $f(q) \leq f(x)$ for all $x \in I$.*

As an example, we see that any polynomial f of even degree with positive senior coefficient attains its minimal value: since $f(x) \longrightarrow +\infty$ as $x \longrightarrow \infty$, there exists $q \in \mathbb{R}$ such that $f(q) \leq f(x)$ for all $x \in \mathbb{R}$.

4.5. Uniformly continuous functions

A function $f \colon A \longrightarrow \mathbb{R}$ is said to be *uniformly continuous on A* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in A$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. (By definition, f is continuous on A if for any $x \in A$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $y \in A$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$; the difference is that in the case of uniform continuity, δ depends on ε only and does not depend on x.)

f is not uniformly continuous on A if there is $\varepsilon > 0$ such that for every $\delta > 0$ there are $x, y \in A$ with $|x - y| < \delta$ such that $|f(x) - f(y)| \geq \varepsilon$; equivalently, if there are two sequences (x_n) and (y_n) in A such that $x_n - y_n \longrightarrow 0$ but $f(x_n) - f(y_n) \not\longrightarrow 0$.

Examples. (i) The function $f(x) = 2x$ is uniformly continuous on $\mathbb{R}$: given $\varepsilon > 0$, $\delta = \varepsilon/2$ works.
(ii) The function $f(x) = x^2$ is continuous but not uniformly cotinuous on $\mathbb{R}$: $(n + 1/n) - n \longrightarrow 0$, but $f(n + 1/n) - f(n) = (n + 1/n)^2 - n^2 = 2 + 1/n^2 > 2$ for all n.

(iii) The function $f(x) = 1/x$ is continuous but not uniformly continuous on $(0,1]$: $\frac{1}{n} - \frac{1}{n+1} \longrightarrow 0$, but $f(1/(n+1)) - f(1/n) = (n+1) - n = 1$ for all n.

A function $f\colon A \longrightarrow \mathbb{R}$ is said to be *Lipschitz* if there is $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in A$. Clearly, any Lipschitz function is uniformly continuous: given $\varepsilon > 0$, put $\delta = \varepsilon/C$, then $|f(x) - f(y)| \leq C|x - y| < \varepsilon$ whenever $|x - y| > \varepsilon/C$. The converse is not true: the function $f(x) = \sqrt{x}$ is uniformly continuous on $[0,1]$ (as Theorem 4.5.1 below says) but is not Lipschitz:

$$\tfrac{f(1/n)-f(1/(n+1))}{1/n-1/(n+1)} = \tfrac{\sqrt{1/n}-\sqrt{1/(n+1)}}{1/n-1/(n+1)} = \tfrac{\sqrt{1/n}-\sqrt{1/(n+1)}}{\left(\sqrt{1/n}-\sqrt{1/(n+1)}\right)\left(\sqrt{1/n}+\sqrt{1/(n+1)}\right)} = \tfrac{1}{(1/\sqrt{n})+(1/\sqrt{n+1})} \longrightarrow +\infty$$

as $n \longrightarrow \infty$, so there is no C such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y > 0$.

On a closed bounded interval any continuous function is uniformly continuous:

Theorem 4.5.1. *Let I be a closed bounded interval and let $f\colon I \longrightarrow \mathbb{R}$ be continuous. Then f is uniformly continuous.*

Proof. Suppose that f is not uniformly continuous, that is, there is $\varepsilon > 0$ such that for any $\delta > 0$ there are $x, y \in I$ with $|x - y| < \delta$ such that $|f(x) - f(y)| \geq \varepsilon$. We will use the compactness of I, that is, the Bolzano-Weierstrass theorem. For every $n \in \mathbb{N}$ find $x_n, y_n \in I$ such that $|x_n - y_n| < 1/n$ but $\left|f(x_n) - f(y_n)\right| \geq \varepsilon$. Choose a converging subsequence (x_{n_i}) of (x_n), let $x_{n_i} \longrightarrow p$, then, since I is closed, $p \in I$. Since $|x_{n_i} - y_{n_i}| < 1/n_i$ for all i, we have $x_{n_i} - y_{n_i} \longrightarrow 0$, so $y_{n_i} \longrightarrow p$ as well. Hence, both $\lim f(x_{n_i}) = \lim f(y_{n_i}) = f(p)$, so $\lim\bigl(f(x_{n_i}) - f(y_{n_i})\bigr) = 0$, which contradicts $|f(x_{n_i}) - f(y_{n_i})| \geq \varepsilon$ for all i. ■

We can now show that the function $f(x) = \sqrt{x}$ is uniformly continuous on $[0, +\infty)$. By Theorem 4.5.1, f is uniformly continuous on $[0,1]$; on $[1, +\infty)$, f is Lipschitz:

$$|f(y) - f(x)| = \left|\sqrt{y} - \sqrt{x}\right| = \tfrac{|y-x|}{\sqrt{y}+\sqrt{x}} \leq \tfrac{1}{2}|y - x|,$$

and so, also uniformly continuous. It follows that f is uniformly continuous on entire $[0, +\infty)$. (By "glueing together" two uniformly continuous functions we get an uniformly continuous function: Given $\varepsilon > 0$, let $\delta_1, \delta_2 > 0$ be such that $|f(y) - f(x)| < \varepsilon/2$ for $x, y \in [0,1]$ with $|y - x| < \delta_1$ or for $x, y \in [1, +\infty)$ with $|y - x| < \delta_2$. Put $\delta = \min\{\delta_1, \delta_2\}$, and let $x, y \geq 0$, $|y - x| < \delta$. Then if both $x, y \leq 1$ or $x, y \geq 1$ we have $|f(y) - f(x)| < \varepsilon/2 < \varepsilon$, and if, say, $x < 1 < y$, then $|x - 1|, |y - 1| < \delta$, so $|f(x) - f(1)|, |f(y) - f(1)| < \varepsilon/2$, so $|f(y) - f(x)| < \varepsilon$.)

A nice property of a uniformly continuous function is that it has a finite limit at every limit point of its domain, and can be continuously extended to these points. We will need the following simple lemma:

Lemma 4.5.2. *Let $f\colon A \longrightarrow \mathbb{R}$ be uniformly continuous on A and let (x_n) be a Cauchy sequence in A; then the sequence $(f(x_n))$ is also Cauchy.*

(If f is continuous and (x_n) is a Cauchy sequence in $\mathrm{Dom}(f)$, then (x_n) converges, and if $\lim x_n = z \in \mathrm{Dom}(f)$, then $f(x_n) \longrightarrow f(z)$; the problem only appears if $z \notin \mathrm{Dom}(f)$.)

Proof. Given $\varepsilon > 0$, find $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ for all $x, y \in A$ with $|y - x| < \delta$. Find k such that $|x_n - x_m| < \delta$ for all $n, m \geq k$, then $|f(x_n) - f(x_m)| < \varepsilon$ for all $n, m \geq k$, so $(f(x_n))$ is Cauchy. ■

For a set $A \subseteq \mathbb{R}$, let A' be the set of limit points of A; the set $\overline{A} = A \cup A'$ is called *the closure* of A. Given functions $f\colon A \longrightarrow Y$ and $\widetilde{f}\colon B \longrightarrow Y$ with $A \subseteq B$, we say that $\widetilde{f}$ is *an extension* of f to B and f is *a restriction* of $\widetilde{f}$ to A and write $f = \widetilde{f}|_A$ if $f(x) = \widetilde{f}(x)$ for all $x \in A$.

The following theorem says that a uniformly continuous function can be continuously extended to the closure of its domain:

Theorem 4.5.3. *Let $f\colon A \longrightarrow \mathbb{R}$ be a uniformly continuous function; then there is a uniformly continuous function $\widetilde{f}\colon \overline{A} \longrightarrow \mathbb{R}$ such that $f = \widetilde{f}|_A$.*

This theorem is not, of course, true for just continuous, non-uniformly continuous functions: the function $f(x) = 1/x$ on $(0,1]$ cannot be extended to a function continuous on $[0,1]$.

Proof. For any $x \in \overline{A}$, choose a sequence (x_n) in A with $x_n \longrightarrow x$. (If x is an isolated point of A, we just put $x_n = x$ for all n.) Then, by Lemma 4.5.2, $(f(x_n))$ is Cauchy, so converges; put $\widetilde{f}(x) = \lim f(x_n)$. I claim that $\widetilde{f}(x)$ doesn't depend on the choice of the sequence (x_n): if (y_n) is another sequence in A that converges to x, then the sequence $(x_1, y_1, x_2, y_2, \ldots)$ also converges to x, so the sequence $\big(f(x_1), f(y_1), f(x_2), f(y_2), \ldots\big)$ converges, so its subsequences $\big(f(x_1), f(x_2), \ldots\big)$ and $\big(f(y_1), f(y_2), \ldots\big)$ converge to the same limit. Also if $x \in A$, then $\widetilde{f}(x) = f(x)$ since as (x_n) converging to x we can take the constant sequence $x_n = x$ for all n. So, $\widetilde{f}$ is an extension of f; we only need to show that it is uniformly continuous on $\overline{A}$.

Let $\varepsilon > 0$. Find $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon/3$ for any $x, y \in A$ with $|y - x| < \delta$. Let $x, y \in \overline{A}$ be such that $|x - y| < \delta/3$. Choose sequences (x_n) and (y_n) in A such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$; then $f(x_n) \longrightarrow \widetilde{f}(x)$ and $f(y_n) \longrightarrow \widetilde{f}(y)$. Find n such that $|x_n - x| < \delta/3$, $|y_n - y| < \delta/3$, $|f(x_n) - \widetilde{f}(x)| < \varepsilon/3$, and $|f(y_n) - \widetilde{f}(y)| < \varepsilon/3$. Then

$$|x_n - y_n| \le |x_n - x| + |x - y| + |y - y_n| < \delta/3 + \delta/3 + \delta/3 = \delta,$$

so $|f(x_n) - f(y_n)| < \varepsilon/3$, so

$$|\widetilde{f}(x) - \widetilde{f}(y)| \le |\widetilde{f}(x) - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - \widetilde{f}(y)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

■

4.6. The exponential, logarithmic, and power functions

Let $a \in \mathbb{R}$, $a > 0$. We defined a^n for $n \in \mathbb{N}$ inductively: $a^1 = a$ and $a^{n+1} = a^n a$ for all n. By induction, for any $n \in \mathbb{N}$, if $a > 1$ then $a^n > 1$, and if $0 < a < 1$ then $0 < a^n < 1$.

We then define $a^0 = 1$ and for $n \in \mathbb{N}$, $a^{-n} = (a^{-1})^n = (a^n)^{-1}$. It follows that for any $n \in \mathbb{Z}$ with $n < 0$, if $a > 1$ then $a^n < 1$, and if $0 < a < 1$ then $a^n > 1$. We proved that for any $n, m \in \mathbb{N}$, $a^{n+m} = a^n a^m$ and $(a^n)^m = a^{nm}$, and this identity can be easily extended to the case where $n, m \in \mathbb{Z}$. (If, say, $n, m \in \mathbb{N}$ with $n \ge m$ then $a^n = a^{n-m}a^m$, so $a^{n+(-m)} = a^{n-m} = a^n(a^m)^{-1} = a^n a^{-m}$, etc.)

Also, for any $a, b \in \mathbb{R}$ and $n \in \mathbb{Z}$, $(ab)^n = a^n b^n$.

For $m \in \mathbb{N}$, define $a^{1/m} = \sqrt[m]{a}$, that is, $a^{1/m} = b > 0$ such that $b^m = a$ (and was proven to exist and be unique). It follows (by contraposition) that for all $m \in \mathbb{N}$, if $a > 1$ then $a^{1/m} > 1$, and if $0 < a < 1$ then $a^{1/m} < 1$.

Now, for any $n \in \mathbb{Z}$ and $m \in \mathbb{N}$ define $a^{n/m} = (a^{1/m})^n$; since $\big((a^{1/m})^n\big)^m = \big((a^{1/m})^m\big)^n = a^n$, we see that also $a^{n/m} = (a^n)^{1/m}$. We check that for $r \in \mathbb{Q}$, a^r is well defined: if $n/m = n'/m'$, then $nm' = n'm$, so $a^{nm'} = a^{n'm}$, and so $\big((a^{1/m})^n\big)^{mm'} = \big((a^{1/m})^m\big)^{nm'} = a^{nm'}$ and $\big((a^{1/m'})^{n'}\big)^{mm'} = \big((a^{1/m'})^{m'}\big)^{n'm} = a^{n'm}$ are equal. So, the function $f(x) = a^x$ is defined on $\mathbb{Q}$.

Next, for any $r, s \in \mathbb{Q}$, $a^r a^s = a^{r+s}$: if $r = n/m$ and $s = n'/m'$, we have

$$\big(a^{n/m}a^{n'/m'}\big)^{mm'} = (a^{n/m})^{mm'}(a^{n'/m'})^{mm'} = a^{nm'}a^{n'm} = a^{nm'+n'm},$$

so $a^{n/m}a^{n'/m'} = \sqrt[mm']{a^{nm'+n'm}} = a^{(nm'+n'm)/mm'} = a^{n/m+n'/m'}$.

And for any $r, s \in \mathbb{Q}$, $(a^r)^s = a^{rs}$: if $r = n/m$ and $s = n'/m'$, we have

$$\big((a^{n/m})^{n'/m'}\big)^{mm'} = (a^{n/m})^{n'm} = a^{nn'},$$

so $(a^{n/m})^{n'/m'} = \sqrt[mm']{a^{nn'}} = a^{(nn')/(mm')}$.

And for any $a, b > 0$ and $r \in \mathbb{Q}$, $(ab)^r = a^r b^r$: for $r = n/m$, $\big(a^{n/m}b^{n/m}\big)^m = (a^{n/m})^m(b^{n/m})^m = a^n b^n = (ab)^n$, so $a^{n/m}b^{n/m} = (ab)^{n/m}$.

If $a > 1$, then the function $f(x) = a^x$ is strictly increasing: for $r = n/m > 0$, $n, m \in \mathbb{N}$, we have $a^r = \sqrt[m]{a^n} > 1$, so if $x, y \in \mathbb{Q}$ and $y > x$, then $a^y = a^x a^{y-x} > a^x$. If $0 < a < 1$, then f is strictly decreasing: as $a^{-1} > 1$, if $x, y \in \mathbb{Q}$ and $y > x$, we have $(a^y)^{-1} = (a^{-1})^y > (a^{-1})^x = (a^x)^{-1}$, so $a^y < a^x$. And if $a = 1$, then $a^z = 1$ for all $x \in \mathbb{Q}$.

We now want to extend function $f(x) = a^x$ from $\mathbb{Q}$ to $\mathbb{R}$. We could do it "by monotonicity", but I'll use Theorem 4.5.3: $\mathbb{Q}$ is dense in $\mathbb{R}$, so $\overline{\mathbb{Q}} = \mathbb{R}$. f is not uniformly continuous on $\mathbb{Q}$; it is however *locally uniformly continuous*:

Lemma 4.6.1. *For any $M > 0$, $f(x) = a^x$ is unformly continuous on $\mathbb{Q} \cap [-M, M]$.*

Proof. If $a = 1$, then f is constant, $f(x) = 1$ for all x. If $0 < a < 1$, we have $f(x) = a^x = (a^{-1})^{-x}$, so $f(x) = g(-x)$ where $g(x) = (a^{-1})^x$ where $a^{-1} > 1$, and f is uniformly continuous on $[-M, M]$ iff g is. Thus, we may and will assume that $a > 1$.

First, we will show that f is continuous at 0. Since f is strictly increasing, both $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$ exist and $\lim_{x\to 0^-} f(x) \le f(0) = 1 \le \lim_{x\to 0^+} f(x)$. As we know, $\lim_{n\to\infty} f(1/n) = a^{1/n} = \sqrt[n]{a} \longrightarrow 1$, so $\lim_{x\to 0^+} f(x) = 1$. Also $f(-1/n) = a^{-1/n} = 1/\sqrt[n]{a} \longrightarrow 1$, so $\lim_{x\to 0^-} f(x) = 1$. So, $\lim_{x\to 0} f(x) = 1 = f(0)$.

Now, let $M > 0$; for any $x, y \in \mathbb{Q} \cap (-\infty, M]$, since $f(x) = a^x \le f(M) = a^M$, we have $|f(y) - f(x)| = |a^y - a^x| = a^x|a^{y-x} - 1| \le a^M|a^{y-x} - 1|$. Given $\varepsilon > 0$ let $\delta > 0$ be such that $|f(z) - 1| = |a^z - 1| < \varepsilon/a^M$ if $|z| < \delta$; then $|f(y) - f(x)| < \varepsilon$ whenever $|y - x| < \delta$. ∎

Hence, for any $M \in \mathbb{N}$, function f extends to a continuous function $\widetilde{f}$ on $[-M, M]$. Hence $\widetilde{f}(x)$ is defined for all $x \in \mathbb{R}$ and is continuous on $\mathbb{R}$; we call it *the exponential function with base* a and denote by $\exp_a$. For $x \in \mathbb{R}$, we now put $a^x = \exp_a x$. (That is, for any $x \in \mathbb{R}$, a^x is defined as the limit of the sequence a^{x_n} where (x_n) is any sequence in $\mathbb{Q}$ that converges to x.)

Theorem 4.6.2. *Let $a > 0$.*
(i) *For any $x, y \in \mathbb{R}$, $a^{x+y} = a^x a^y$.*
(ii) *For any $x \in \mathbb{R}$, $a^{-x} = (a^x)^{-1} = (a^{-1})^x$.*
(iii) *For any $a, b > 0$ and $x \in \mathbb{R}$, $(ab)^x = a^x b^x$.*
(iv) *If $a = 1$, then $a^x = 1$ for all x. If $a \ne 1$, the function a^x is strictly monotone: if $a > 1$, then a^x is strictly increasing with $\lim_{x\to+\infty} a^x = +\infty$ and $\lim_{x\to-\infty} a^x = 0$; if $a < 1$, then a^x is strictly decreasing with $\lim_{x\to+\infty} a^x = 0$ and $\lim_{x\to-\infty} a^x = +\infty$.*

(I don't see how to prove the identity $(a^x)^y = a^{xy}$ right now!)

Proof. (i) We have $a^{x+y} = a^x a^y$ for all $x, y \in \mathbb{Q}$; by continuity, $\exp_a(x + y) = \exp_a x \exp_a y$ for all $x, y \in \mathbb{R}$. Indeed, for $x, y \in \mathbb{R}$, take any sequences (x_n) and (y_n) in $\mathbb{Q}$ such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$, then $x_n + y_n \longrightarrow x + y$, so

$$a^{x+y} = \lim_{n\to\infty} a^{x_n+y_n} = \lim_{n\to\infty} a^{x_n} a^{y_n} = \lim_{n\to\infty} a^{x_n} \lim_{n\to\infty} a^{y_n} = a^x a^y.$$

(ii) For any x, $1 = a^0 = a^{x-x} = a^x a^{-x}$, so $a^{-x} = (a^x)^{-1}$. Let (x_n) be a sequence in $\mathbb{Q}$ with $x_n \longrightarrow x$, then $a^{-x} = \lim_{n\to\infty} a^{-x_n} = \lim_{n\to\infty}(a^{-1})^{x_n} = (a^{-1})^x$.
(iii) Let (x_n) be a sequence in $\mathbb{Q}$ with $x_n \longrightarrow x$, then $(ab)^x = \lim_{n\to\infty}(ab)^{x_n} = \lim_{n\to\infty} a^{x_n} b^{x_n} = a^x b^x$.
(iv) Assume that $a > 1$, then the function a^x is strictly increasing on $\mathbb{Q}$. Let $x < y$; find $s, r \in \mathbb{Q}$ such that $x < s < r < y$, find sequences (x_n) and (y_n) in $\mathbb{Q}$ such that $x_n \longrightarrow x$ with $x_n < s$ for all n and $y_n \longrightarrow y$ with $y_n > r$ for all n. Then $a^{x_n} < a^s$ and $a^{y_n} > a^r$ for all n, so

$$a^x = \lim_{n\to\infty} a^{x_n} \le a^s < a^r \le \lim_{n\to\infty} a^{y_n} = a^y.$$

So, a^x is strictly increasing. It follows that both $\lim_{x\to+\infty} a^x$ and $\lim_{x\to-\infty} a^x$ exist; since $\lim_{n\to\infty} a^n = +\infty$ and $\lim_{n\to\infty} a^{-n} = 0$, we have $\lim_{x\to+\infty} a^x = +\infty$ and $\lim_{x\to-\infty} a^x = 0$.

If $a < 1$, then since $a^x = (a^{-1})^{-x}$ and $a^{-1} > 1$, we have that $(a^{-1})^x$ is strictly increasing and a^x is strictly decreasing with $\lim_{x\to+\infty} a^x = 0$ and $\lim_{x\to-\infty} a^x = +\infty$. ∎

Since, for $a \ne 1$, $\exp_a$ is strictly monotone, it has an inverse, which is called *the logarithmic function* with base a and is denoted by $\log_a$, so that $\exp_a x = a^x = y$ iff $y = \log_a x$. (Or, in other words, for any $x \in \mathbb{R}$, $\log_a a^x = x$, and for any $x > 0$, $a^{\log_a x} = x$.)

Theorem 4.6.3. *Let $a > 0$, $a \ne 1$.*
(i) *The function $\log_a$ is continuous with $\mathrm{Dom}(\log_a) = \mathrm{Rng}(\exp_a) = (0, +\infty)$.*
(ii) *If $a > 1$ then $\log_a$ is strictly increasing with $\lim_{x\to 0^+} \log_a x = -\infty$ and $\lim_{x\to+\infty} \log_a x = +\infty$; if $a < 1$ then $\log_a$ is strictly decreasing with $\lim_{x\to 0^+} \log_a x = +\infty$ and $\lim_{x\to+\infty} \log_a x = -\infty$.*

(iii) *For any* $x, y > 0$, $\log_a(xy) = \log_a x + \log_a y$.

Proof. (i) follows from the definition and Theorem 4.3.6.
(ii) The inverse of a strictly increasing function is strictly increasing (if $y_1 < y_2$ implies that $x_1 = f^{-1}(y_1) < x_2 = f^{-1}(y_2)$, since otherwise $y_1 = f(x_1) \geq y_2 = f(x_2)$), and of strictly decreasing function is strictly decreasing. And $\mathrm{Rng}(\log_a) = \mathrm{Dom}(\exp_a) = (-\infty, +\infty)$.
(iii) For any $x, y > 0$, $a^{\log_a x + \log_a y} = a^{\log_a x} a^{\log_a y} = xy$, so $\log_a(xy) = \log_a x + \log_a y$. ■

Now, when we know what a^x is for all $a > 0$ and all $x \in \mathbb{R}$, given any $\alpha \in \mathbb{R}$ we have another function, *the power function* $x^\alpha = \exp_x \alpha$, with domain $(0, +\infty)$. (This function is defined as "a limit" of power functions with rational exponents: we choose any sequence (α_n) in $\mathbb{Q}$ with $\alpha_n \longrightarrow \alpha$, then for any $x > 0$, $x^\alpha = \lim_{n\to\infty} x^{\alpha_n}$.)

Theorem 4.6.4. *Let* $\alpha \in \mathbb{R}$.
(i) *For any* $x, y > 0$, $(xy)^\alpha = x^\alpha y^\alpha$.
(ii) *For any* $x > 0$, $x^{-\alpha} = (1/x)^\alpha = (x^\alpha)^{-1}$.
(iii) *If* $\alpha > 0$, *the function* x^α *is strictly increasing with* $\lim_{x\to 0^+} x^\alpha = 0$ *and* $\lim_{x\to+\infty} x^\alpha = +\infty$; *if* $\alpha < 0$, *the function* x^α *is strictly decreasing with* $\lim_{x\to 0^+} x^\alpha = +\infty$ *and* $\lim_{x\to+\infty} x^\alpha = 0$.
(iv) *The function* x^α *is continuous.*

Proof. (i) This is Theorem 4.6.2(iii). (ii) This is Theorem 4.6.2(ii).
(iii) Let $\alpha > 0$. For any $z > 1$, since the function $f(x) = z^x$ is strictly increasing, $z^\alpha > 1$ since $\alpha > 0$. So, for any $0 < x < y$, since $y/x > 1$, we have $y^\alpha/x^\alpha = (y/x)^\alpha > 1$, so $y^\alpha > x^\alpha$.

Thus the function x^α is strictly increasing on $(0, +\infty)$, and so, $\lim_{x\to 0^+} x^\alpha$ and $\lim_{x\to+\infty} x^\alpha$ exist. Now, let $n \in \mathbb{N}$ be such that $1/n < \alpha$, then for all $x > 1$, $x^\alpha > x^{1/n}$, and $\lim_{x\to+\infty} x^{1/n} = +\infty$, so $\lim_{x\to+\infty} x^\alpha = +\infty$. Also,

$$\lim_{x\to 0^+} (1/x^\alpha) = \lim_{x\to 0^+} (1/x)^\alpha = \lim_{x\to+\infty} x^\alpha = +\infty,$$

so $\lim_{x\to 0^+} x^\alpha = 0$.

If $\alpha < 0$, we can write $x^\alpha = (x^{-1})^{-\alpha}$. So, x^α is strictly decreasing (if $0 < x < y$, then $x^{-1} > y^{-1}$, so $x^\alpha = (x^{-1})^{-\alpha} > (y^{-1})^{-\alpha} = y^\alpha$), $\lim_{x\to 0^+} x^\alpha = \lim_{x\to+\infty} x^{-\alpha} = +\infty$, and $\lim_{x\to+\infty} x^\alpha = \lim_{x\to 0^+} x^{-\alpha} = 0$.
(iv) Firstly I claim that if the function $h(x) = x^\alpha$ is continuous at some point $z > 0$ then it is continuous at every other point $x > 0$. Indeed, let (x_n) be a sequence that converges to x. Then $x_n z/x \longrightarrow z$, so $x_n^\alpha z^\alpha / x^\alpha = (x_n z/x)^\alpha \longrightarrow z^\alpha$, so $x_n^\alpha \longrightarrow x^\alpha$. Hence, h is continuous at x.

Now, $h(x) = x^\alpha$ is a monotone function on $(0, +\infty)$, so it has at most countably many points of discontinuity, so it is continuous at some (in fact, uncountably many) points. But then it is continuous at all points, hence is continuous. ■

For $\alpha > 0$, since $\lim_{x\to 0^+} x^\alpha = 0$, we may extend this function by continuity to 0, and put $0^\alpha = 0$.

We can now prove "the second" exponential identity:

Theorem 4.6.5. *Let* $a > 0$. *For any* $x, y \in \mathbb{R}$, $(a^x)^y = a^{xy}$.

Proof. First, let $x \in \mathbb{Q}$, $y \in \mathbb{R}$, and let (y_n) be a sequence in $\mathbb{Q}$ with $y_n \longrightarrow y$; then for any n, $(a^x)^{y_n} = a^{xy_n}$, so

$$(a^x)^y = \lim_{n\to\infty} (a^x)^{y_n} = \lim_{n\to\infty} a^{xy_n} = a^{xy}.$$

Now, let both $x, y \in \mathbb{R}$, and let (x_n) be a sequence in $\mathbb{Q}$ with $x_n \longrightarrow x$. Then since the function $h(z) = z^y$ is continuous and $a^{x_n} \longrightarrow a^x$, we have $\lim_{n\to\infty} (a^{x_n})^y = (a^x)^y$. On the other hand, since for any n, $x_n \in \mathbb{Q}$, we have $(a^{x_n})^y = a^{x_n y}$ as proved above, so $\lim_{n\to\infty} (a^{x_n})^y = \lim_{n\to\infty} a^{x_n y} = a^{xy}$. ■

It follows that for any $\alpha \neq 0$, the inverse of the function $h(x) = x^\alpha$ is $h^{-1}(x) = x^{1/\alpha}$.

As a corollary, we also obtain relations
(i) between exponential and power functions: for any $a > 0$, $a \neq 1$, any $\alpha \in \mathbb{R}$, and any $x > 0$, since $x = a^{\log_a x}$, we have $x^\alpha = (a^{\log_a x})^\alpha = a^{\alpha \log_a x}$;
(ii) between exponential functions of various bases: for any $a, b > 0$, $a \neq 1$, since $b = a^{\log_a b}$, for any $x \in \mathbb{R}$ we have $b^x = (a^{\log_a b})^x = a^{x \log_a b} = a^{cx}$ where $c = \log_a b$;

(iii) and between logarithmic functions of various bases: from (ii), for any $a, b > 0$, $a, b \neq 1$, we have $\exp_b y = \exp_a(cy)$ where $c = \log_a b$, so $y = \log_a(\exp_b y)/\log_a b$. Taking $y = \log_b x$ we get $\log_b x = \log_a x/\log_a b$.

Let us now show that the linear, exponential, logarithmic, and power functions are the only continuous functions satisfying the corresponding functional equations. Let's start with the linear functions:

Theorem 4.6.6. *If $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function satisfying $f(x+y) = f(x) + f(y)$ for all x and y, then $f(x) = cx$ for some $c \in \mathbb{R}$.*

Proof. By induction on n, for any $a \in \mathbb{R}$ and any $n \in \mathbb{N}$, $f(na) = nf(a)$. (Indeed, if this is true for some n, then $f((n+1)a) = f(na + a) = f(na) + f(a) = nf(a) + f(a) = (n+1)f(a)$.)

Now put $c = f(1)$. Then for any $n \in \mathbb{N}$, $f(n) = f(n \cdot 1) = nf(1) = nc$.

Next, for any $m \in \mathbb{N}$, $c = f(1) = f\big(m\frac{1}{m}\big) = mf\big(\frac{1}{m}\big)$, so $f\big(\frac{1}{m}\big) = \frac{c}{m}$. Also, for any $n, m \in \mathbb{N}$, $f\big(\frac{n}{m}\big) = nf\big(\frac{1}{m}\big) = n\frac{c}{m} = c\frac{n}{m}$. So, $f(x) = cx$ for all positive $x \in \mathbb{Q}$.

Next, $f(0) = f(0+0) = f(0) + f(0) = 2f(0)$, so $f(0) = 0$. Thus, for any positive $x \in \mathbb{Q}$, $f(-x) + f(x) = f(-x + x) = f(0) = 0$, so $f(-x) = -f(x) = -cx$. Hence, $f(x) = cx$ for all $x \in \mathbb{Q}$.

Finally, we have two continuous functions, f and cx, that coincide on the dense set $\mathbb{Q}$. This implies that $f(x) = cx$ for all x. ∎

Remark. The assertion of the theorem doesn't remain true if we drop the assumption that f is continuous: there are many everywhere(!) discontinuous functions f satisfying $f(x+y) = f(x) + f(y)$ for all x, y.

Theorem 4.6.7. *If $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a nonzero continuous function satisfying $f(x+y) = f(x)f(y)$ for all x and y, then $f = \exp_a$ for some $a > 0$.*

Proof. We can mimic the argument used in the proof of the previous theorem, but also can reduce the problem to that theorem. First of all, for any $x \in \mathbb{R}$, $f(x) = f(x/2)f(x/2) = f(x/2)^2 \geq 0$. Next, if $f(x) = 0$ for some x, then $f(z) = f(x)f(z-x) = 0$ for all z; since $f \neq 0$, it must be that $f(x) > 0$ for all x.

Now take any $b > 0$ and define $g(x) = \log_b f(x)$, $x \in \mathbb{R}$. Then for any $x, y \in \mathbb{R}$,

$$g(x+y) = \log_b f(x+y) = \log_b(f(x)f(y)) = \log_b f(x) + \log_b f(y) = g(x) + g(y).$$

g is a composition of continuous functions, thus is continuous, so by Theorem 4.6.6, $g(x) = cx$ for some c. Hence, $f(x) = b^{g(x)} = b^{cx} = (b^c)^x = a^x$ for $a = b^c$. ∎

Theorem 4.6.8. *If $f\colon (0, +\infty) \longrightarrow \mathbb{R}$ is a nonzero continuous function satisfying $f(xy) = f(x) + f(y)$ for all $x, y > 0$, then $f = \log_a$ for some $a > 0$, $a \neq 1$.*

Proof. Take any $b > 0$ and define $g(x) = f(b^x)$, $x \in \mathbb{R}$. Then for any $x, y \in \mathbb{R}$ we have

$$g(x+y) = f(b^{x+y}) = f(b^x b^y) = f(b^x) + f(b^y) = g(x) + g(y).$$

g is a composition of continuous functions, so is continuous, so by Theorem 4.6.6, $g(x) = cx$ for some c. Thus, $f(x) = g(\log_b x) = c\log_b x$. Since $f \neq 0$, $c \neq 0$.

Now put $a = b^{1/c}$, so that $b = a^c$ and $c = \log_a b$. Then $f(x) = c\log_b x = \log_a b \log_b x = \log_a x$. ∎

Theorem 4.6.9. *If $f\colon (0, +\infty) \longrightarrow \mathbb{R}$ is a nonzero continuous function satisfying $f(xy) = f(x)f(y)$ for all $x, y > 0$, then $f(x) = x^\alpha$ for some $\alpha \in \mathbb{R}$.*

Proof. For any $x > 0$, $f(x) = f(\sqrt{x})^2 \geq 0$. If $f(x) = 0$ for some x, then $f(z) = f(x)f(z/x) = 0$ for all z. So, $f(x) > 0$ for all x.

Take any $b > 0$ and define $g(x) = \log_b f(x)$. Then for any $x, y > 0$,

$$g(xy) = \log_b f(xy) = \log_b f(x)f(y) = \log_b f(x) + \log_b f(y) = g(x) + g(y).$$

By Theorem 4.6.8, $g(x) = \log_a x$ for some $a > 0$. Let $\alpha = \log_a b$, so that $b = a^\alpha$. Then $f(x) = b^{g(x)} = (a^\alpha)^{\log_a x} = (a^{\log_a x})^\alpha = x^\alpha$. ∎

5. Derivatives

5.1. The derivative of a function at a point

Let function f be defined in a neighborhood of a point a. We say that f is *differentiable* at a if a finite limit $c = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists; this is the case, the number c is called *the derivative of* f at a and is denoted by $f'(a)$. Equivalently, $c = f'(a)$ can be defined as $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.

f is said to be *left-hand differentiable* at a if f is defined on an interval $(b, a]$ and a finite limit $c = \lim_{x\to a^-} \frac{f(x)-f(a)}{x-a}$ exists; c is called *the left-hand derivative* of f at a and is denoted by $f'_-(a)$. f is said to be *right-hand differentiable* at a if f is defined on an interval $[a, b)$ and $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} = c$; c is called *the right-hand derivative* of f at a and is denoted by $f'_+(a)$.

The "little o" and "big O" notations are very handy. For two functions φ and ψ we write $\varphi(x) = O(\psi(x))$ and say that φ *is O-big of* ψ as $x \longrightarrow \alpha$ (where α can be any of a, $a^\pm$ with $a \in \mathbb{R}$, $\pm\infty$, or ∞) if φ/ψ is bounded in a neighborhood of α, that is, if there is $C > 0$ such that $|\varphi(x)| \le C|\psi(x)|$ for all x in a neighborhood of α. We write $\varphi(x) = o(\psi(x))$ and say that φ *is o-small of* ψ as $x \longrightarrow \alpha$ if $\varphi/\psi \longrightarrow 0$ as $x \longrightarrow \alpha$. (For example, $x^2 = o(x)$ as $x \longrightarrow 0$, and $2^{-x} = o(x^{-2})$ as $x \longrightarrow +\infty$.)

A function f is differentiable at a point a iff it is "well approximable" near a by a linear function:

Theorem 5.1.1. *Let f be defined in a neighborhood of a; then f is differentiable at a iff there is $c \in \mathbb{R}$ such that $f(x) = f(a) + c(x-a) + o(x-a)$ as $x \longrightarrow a$ (equivalently, $f(a+h) = f(a) + ch + o(h)$ as $h \longrightarrow 0$), in which case $f'(a) = c$.*

Proof. Assume that f is differentiable at c with $f'(a) = c$, that is, $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = c$. Define $\varphi(x) = f(x) - f(a) - c(x-a)$, then $\varphi(x)/(x-a) = \frac{f(x)-f(a)}{x-a} - c \longrightarrow 0$ as $x \longrightarrow a$, that is, $\varphi(x) = o(x-a)$ as $x \longrightarrow a$.

Conversely, let $f(x) = f(a) + c(x-a) + \varphi(x)$ with $\varphi(x)/(x-a) \longrightarrow 0$ as $x \longrightarrow a$. Then $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = c + \lim_{x\to a} \frac{\varphi(x)}{x-a} = c$. ∎

Examples. (i) Let f be a constant function, $f(x) = b$ for all x. Then $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{b-b}{x-a} = 0$ for all a.

(ii) Let f be a linear function, $f(x) = bx$, $x \in \mathbb{R}$. Then $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{bx-ba}{x-a} = b$ for any a. (Alternatively, $f(a+h) = ba + bh = f(a) = bh$, so $f'(a) = b$.)

(iii) Let $f(x) = x^2$, $x \in \mathbb{R}$. Let $a \in \mathbb{R}$. Then $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^2-a^2}{x-a} = \lim_{x\to a}(x+a) = 2a$. (Alternatively, $f(a+h) = (a+h)^2 = a^2 + 2ah + h^2 = f(a) + 2ah + o(h)$, so $f'(a) = 2a$.)

A function $f\colon A \longrightarrow \mathbb{R}$ is said to be *Lipschitz at a point* $a \in A$ if there is a neighborhood I of a and a constant $C > 0$ such that $|f(x) - f(a)| \le C|x-a|$ for all $x \in A \cap I$. Clearly, if f is Lipschitz at a, then f is continuous at a.

Theorem 5.1.2. *If f is differentiable at a point a, then f is Lipschitz (and so, continuous) at a.*

Proof. Since a finite $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ eists, the function $\frac{f(x)-f(a)}{x-a}$ is bounded in a neighborhood I of a, $\left|\frac{f(x)-f(a)}{x-a}\right| \le C$ for all $I \cap A \setminus \{a\}$, so $|f(x) - f(a)| \le C|x-a|$ for all $x \in A \cap I$. ∎

The "squeeze theorem" for derivatives is sometimes helpful:

Theorem 5.1.3. *Suppose that f, g and h are defined in a neighborhood of a, $f(a) = h(a)$, f and h are differentiable at a with $f'(a) = h'(a)$, and $f(x) \le g(x) \le h(x)$ or $h(x) \le g(x) \le f(x)$ for all x in a neighborhood of a. Then g is differentiable at a with $g'(a) = f'(a)$.*

The "algebraic properties" of derivatives are the following:

Theorem 5.1.4. *Let functions f and g be differentiable at a. Then $f+g$ is differentiable at a with $(f+g)'(a)=f'(a)+g'(a)$; for any $b\in\mathbb{R}$, bf is differentiable at a with $(bf)'(a)=bf'(a)$); fg is differentiable at a with $(fg)'(a)=f'(a)g(a)+f(a)g'(a)$; and if $f(a)\neq 0$, then g/f is differentiable at a with $(g/f)'(a)=\big(g'(a)f(a)-g(a)f'(a)\big)/f(a)^2$.*

Proof.

$$\lim_{x\to a}\frac{(f+g)(x)-(f+g)(a)}{x-a}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}+\lim_{x\to a}\frac{g(x)-g(a)}{x-a}=f'(a)+g'(a)$$

$$\lim_{x\to a}\frac{(bf)(x)-(bf)(a)}{x-a}=b\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=bf'(a)$$

$$\begin{aligned}\lim_{x\to a}\frac{(fg)(x)-(fg)(a)}{x-a}&=\lim_{x\to a}\frac{f(x)g(x)-f(a)g(x)}{x-a}+\lim_{x\to a}\frac{f(a)g(x)-f(a)g(a)}{x-a}\\&=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\lim_{x\to a}g(x)+f(a)\lim_{x\to a}\frac{g(x)-g(a)}{x-a}=f'(a)g(a)+f(a)g'(a)\end{aligned}$$

$$\lim_{x\to a}\frac{(1/f)(x)-(1/f)(a)}{x-a}=\lim_{x\to a}\frac{f(a)-f(x)}{x-a}\cdot\frac{1}{f(x)f(a)}=\lim_{x\to a}\frac{f(a)-f(x)}{x-a}\lim_{x\to a}\frac{1}{f(x)f(a)}=\frac{-f'(a)}{f(a)^2}$$

$$\text{and } (g/f)'(a)=g'(a)(1/f)(a)+g(a)(1/f)'(a)=\frac{g'(a)}{f(a)}-\frac{g(a)f'(a)}{f(a)^2}=\frac{g'(a)f(a)-g(a)f'(a)\big)}{f(a)^2}$$

(Notice that if $f(a)\neq 0$ and f is differentiable (and so continuous) at a then $f(x)\neq 0$ in a neighborhood of a, so $1/f(x)$ is deined in this neighborhood.) ■

The following theorem about the derivative of composition is called *the chain rule* for derivative:

Theorem 5.1.5. *Let function f be differentiable at a and function g be differentiable at $f(a)$. Then $g\circ f$ is differentiable at a with $(g\circ f)'(a)=g'(f(a))f'(a)$.*

Proof. First of all, since f is defined in a neighborhood of a, g is defined in a neighborhood of $f(a)$, and f is continuous at a, $g\circ f$ is defined in a neighborhood of a.

Let $b=f(a)$. We have $\frac{g(f(x))-g(f(a))}{x-a}=\frac{g(f(x))-g(f(a))}{f(x)-f(a)}\frac{f(x)-f(a)}{x-a}$, which tends to $g'(f(a))f'(a)$ as $x\longrightarrow a$, but is not defined at the points x where $f(x)=f(a)$. To fix this, let $b=f(a)$ and define function $h(y)=\begin{cases}\frac{g(y)-g(b)}{y-b}, & y\neq b\\ g'(b), & y=b\end{cases}$. Then h is continuous at b, and for any $x\in\mathrm{Dom}(g\circ f)$,

$$\frac{g(f(x))-g(f(a))}{x-a}=h(f(x))\frac{f(x)-f(a)}{x-a}.$$

Since h is continuous at $b=f(a)$ and f is continuous at a, $h\circ f$ is continuous at a, so

$$\lim_{x\to a}h(f(x))\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}h(f(x))\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=h(f(a))f'(a)=g'(f(a))f'(a).$$

■

If f is invertible in a neighborhood of a point a and is differentiable at a, then $f^{-1}\circ f(x)=x$, so by the chain rule, $(f^{-1})'(f(a))f'(a)=x'|_{x=a}=1$, so $(f^{-1})'(f(a))=1/f'(a)$, – ok, but this only works if f^{-1} is diiferentiable at $f(a)$! And actually, f^{-1} may be not only non-differentiable, but discontinuous at $f(a)$. We however have:

Theorem 5.1.6. *Let an invertible function f be differentiable at a with $f'(a)\neq 0$ and let f^{-1} be defined in a neighborhood of $b=f(a)$ and continuous at b. Then f^{-1} is differentiable at b with $(f^{-1})'(b)=1/f'(a)$.*

The condition "f^{-1} is defined in a neighborhood of $b=f(a)$ and continuous at b" holds if f is continuous in a neighborhood I of a, since, as we know, in this case f is strictly monotone on I and f^{-1} is continuous on the open interval $f(I)$ containing b.

Proof. Let $g = f^{-1}$. Since g is bijective, $g(y) \neq g(b) = a$ for all $y \neq b$. Let (y_n) be a sequence that converges to b and such that $y_n \neq b$ for all n. Then the sequence $x_n = g(y_n)$ converges to a and $x_n \neq a$ for all n, and we have

$$\frac{g(y_n) - g(b)}{y_n - b} = \frac{x_n - a}{f(x_n) - f(a)} = \frac{1}{\frac{f(x_n)-f(a)}{x_n-a}} \longrightarrow \frac{1}{f'(a)}.$$

So, $g'(b) = 1/f'(a)$. ■

If a function f is differentiable at all points of a set $A \subseteq \mathbb{R}$ we say that *f is diiferetiable on A*; if $A = \mathrm{Dom}(f)$, we just say that *f is differentiable*. If f is differentiable on A we obtain the function $f' \colon A \longrightarrow \mathbb{R}$ called *the derivative* of f. By theorems above we get:

Theorem 5.1.7. *If f and g are differentiable on a set A, then $f+g$ and fg are differentiable on A with $(f+g)' = f'+g'$, $(fg)' = f'g+fg'$, and f/g is differentiable on $A \setminus \{x : g(x) = 0\}$ with $(f/g)' = (f'g-fg')/g^2$. If f and g are such that $\mathrm{Rng}(f) \subseteq \mathrm{Dom}(g)$, f is differentiable on A and g is differentiable on $f(A)$, then $g \circ f$ is differentiable on A with $(g \circ f)' = (g' \circ f)f'$. If f is differentiable and invertible on A $f'(x) \neq 0$ for all $x \in A$, and f^{-1} is continuous on $f(A)$, then f^{-1} is differentiable on $f(A)$ with $(f^{-1})' = 1/(f' \circ f^{-1})$. If f is differentiable and invertible on an interval I and $f'(x) \neq 0$ for all $x \in I$, then f^{-1} is differentiable on $f(I)$ with $(f^{-1})' = 1/(f' \circ f^{-1})$.*

This theorem allows to easily find the derivatives of functions constructed from "basic" functions, whose derivatives are already known. Let's find the derivatives of the power, exponential, and logarithmic functions:

(i) Let $n \in \mathbb{N}$, $f(x) = x^n$. Then it is easy to see by induction that for any $a \in \mathbb{R}$, $f'(x) = nx^{n-1}$. Indeed, this is true for $n = 1$, and if this is true for some n, then

$$(x^{n+1})' = (x^n x)' = (x^n)'x + ax^n x' = nx^{n-1}x + x^n 1 = (n+1)x^n.$$

(Alternatively, by the binomial formula, for any $x \in \mathbb{R}$,

$$(x+h)^n = x^n + nx^{n-1}h + \tbinom{n}{2}x^{n-2}h^2 + \cdots + h^n = x^n + nx^{n-1}h + o(h),$$

so $(x^n)' = nx^{n-1}$.)

(ii) For $n \in \mathbb{N}$, $(x^{-n})' = (1/x^n)' = -nx^{n-1}/(x^n)^2 = -nx^{-n-1}$.

(iii) For any $n \in \mathbb{N}$, for any $x > 0$,

$$(\sqrt[n]{x})' = \frac{1}{(y^n)'|_{y=\sqrt[n]{x}}} = \frac{1}{n(\sqrt[n]{x})^{n-1}} = \frac{1}{nx^{1-1/n}} = \tfrac{1}{n}x^{1/n-1}.$$

(iv) Let $a > 0$ and $f(x) = a^x$, $x \in \mathbb{R}$. Then for any $x \in \mathbb{R}$, $\frac{f(x+h)-f(x)}{h} = \frac{a^{x+h}-a^x}{h} = a^x\frac{a^h-1}{h}$, so $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ exists iff $f'(0) = \lim_{h\to 0}\frac{a^h-1}{h}$ exists, and then $f'(x) = a^x f'(0) = f'(0)f(x)$. Hence, if $f(x) = a^x$ is differentiable at one point then it is differentiable at 0, then it is differentiable at all points, and $f'(x) = cf(x)$ where $c = f'(0)$. We will prove that f is differentiable in subsection 5.2.

(v) Let $a > 0$, $a \neq 1$, and let $g(x) = \log_a x$, then $g = f^{-1}$ where $f(y) = a^y$, and $(g)'(x) = 1/f'(g(x)) = 1/(cf(g(x))) = 1/(cx)$ where $c = f'(0)$.

(vi) Let $\alpha \in \mathbb{R}$, let $h(x) = x^\alpha$, $x > 0$. Take any $a > 0$, $a \neq 1$; we can write h as the composition $h(x) = x^\alpha = (a^{\log_a x})^\alpha = a^{\alpha\log_a x}$, so by the chain rule, $h'(x) = ca^{\alpha\log_a x} \cdot \alpha \cdot (1/(cx)) = cx^\alpha\alpha/cx = \alpha x^{\alpha-1}$.

5.2. Convex functions

Let I be an open interval (bounded or unbouded). A function $f\colon I \longrightarrow \mathbb{R}$ is said to be *convex* if for any $x, y, z \in I$ with $x < y < z$ we have

$$f(y) \leq f(x) + \frac{f(z) - f(x)}{z - x}(y - x). \tag{5.1}$$

(That is, $f(y) < l(y)$ where $l(y)$ is the linear function $l(y) = f(x) + \frac{f(z)-f(x)}{z-x}(y-x)$ having the property $l(x) = f(x)$ and $l(z) = f(z)$.) f is said to be *strictly convex* if for any $x < y < z$, $f(y) < f(x) + \frac{f(z)-f(x)}{z-x}(y-x)$; *concave* if $f(y) \geq f(x) + \frac{f(z)-f(x)}{z-x}(y-x)$, and strictly concave if $f(y) > f(x) + \frac{f(z)-f(x)}{z-x}(y-x)$.

It is easy to check that inequality (5.1) is equivalent to each of

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x}, \qquad \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}, \qquad \frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(y)}{z - y}. \tag{5.2}$$

Let $f\colon I \longrightarrow \mathbb{R}$ be convex; fix $a \in I$ and consider the function $\varphi_a(x) = \frac{f(x)-f(a)}{x-a}$ on $I \setminus \{a\}$. It follows from inequalities (5.2) that φ_a is increasing; hence, $f'_-(a) = \lim_{x\to a^-} \varphi_a(x)$ and $f'_+(a) = \lim_{x\to a^+} \varphi_a(x)$ both exist and $f'_-(a) \leq f'_+(a)$. It follows that f is continuous at a; so, f is continuous on I.

Since $f'_-(x) \in \mathbb{R}$ and $f'_+(x) \in \mathbb{R}$ are defined for every $x \in I$, f'_- and f'_+ are functions $I \longrightarrow \mathbb{R}$. For any $a, b \in I$ with $a < b$, $f'_-(a) \leq f'_+(a) \leq \varphi_a(b) = \varphi_b(a) \leq f'_-(b) \leq f'_+(b)$. Hence, both f'_- and f'_+ are increasing functions; thus they have at most countably many points of discontinuity. Moreover, for any a, we have that $f'_-(a) \leq f'_+(a) \leq \lim_{b\to a^+} f'_-(b)$, so if f'_- is continuous at a, that is, $f'_-(a) = \lim_{b\to a^+} f'_-(b)$, then $f'_-(a) = f'_+(a)$, that is, f is differentiable at a. We have proved the following theorem:

Theorem 5.2.1. *Let I be an open interval, and let $f\colon I \longrightarrow \mathbb{R}$ be a convex function. Then*
(i) *f is continuous;*
(ii) *the left-hand f'_- and the right-hand f'_+ derivatives exist at every point of I and are increasing functions on I;*
(iii) *f is differentiable at all but at most countably many points of I and f' is an increasing function on its domain.*

A similar theorem holds for concave functions, with f'_-, f'_+ and f' being decreasing functions.

Given an interval $[x, z]$, every point $y \in [x, z]$ is uniquely representable in the form $y = (1-t)x + tz$ for some $t \in [0, 1]$. (The function $t \mapsto (1-t)x + tz$ is a bijection between $[0, 1]$ and $[x, z]$.) Given a function f on an interval $I \supseteq [x, z]$, for $y = (1-t)x + tz$ we have

$$f(x) + \frac{f(z) - f(x)}{z - x}(y - x) = f(x) + (f(z) - f(x))\frac{x - tx + tz - x}{z - x} = f(x) + (f(z) - f(x))t = (1-t)f(x) + tf(z).$$

Hence, f is convex iff for any $t \in [0, 1]$,

$$f\big((1-t)x + tz\big) \leq (1-t)f(x) + tf(z). \tag{5.3}$$

If f is continuous, it suffices to check (5.3) for $t = \frac{1}{2}$ only:

Theorem 5.2.2. *Let f be a continuous function on an open interval I satisfying $f\big(\frac{x+z}{2}\big) \leq \frac{f(x)+f(z)}{2}$ for all $x, z \in I$. Then f is convex on I.*

Proof. Fix $x, z \in I$. We have $f\big(\frac{1}{2}(x+z)\big) \leq \frac{1}{2}\big(f(x) + f(z)\big)$, that is, the inequality (5.3) holds for f for $t = \frac{1}{2}$. Next we have

$$f\big(\tfrac{1}{2}\big(x + \tfrac{1}{2}(x+z)\big)\big) \leq \tfrac{1}{2}\big(f(x) + f\big(\tfrac{1}{2}(x+z)\big)\big) \leq \tfrac{1}{2}\big(f(x) + \tfrac{1}{2}\big(f(x) + f(z)\big)\big),$$

so $f\big(\frac{3}{4}x + \frac{1}{4}z\big) \leq \frac{3}{4}f(x) + \frac{1}{4}f(z)$. Similarly,

$$f\big(\tfrac{1}{2}\big(\tfrac{1}{2}(x+z) + z\big)\big) \leq \tfrac{1}{2}\big(f\big(\tfrac{1}{2}(x+z)\big) + f(z)\big) \leq \tfrac{1}{2}\big(\tfrac{1}{2}\big(f(x) + f(z)\big) + f(z)\big),$$

so $f\big(\frac{1}{4}x + \frac{3}{4}z\big) \leq \frac{1}{4}f(x) + \frac{3}{4}f(z)$. Hence, the inequality (5.3) holds for $t = \frac{1}{4}$ and $t = \frac{3}{4}$. By induction, we can prove in the same way that (5.3) holds for all $t \in [0, 1]$ of the form $t = \frac{k}{2^n}$ with $k, n \in \mathbb{N}$. The set $\big\{\frac{k}{2^n},\ k, n \in \mathbb{N}\big\}$ is dense in $[0, 1]$; by continuity, (5.3) holds for all $t \in [0, 1]$. ■

Now, let $a > 0$, and consider the function $f(x) = \exp_a(x) = a^x$, $x \in \mathbb{R}$. For any $x, z \in \mathbb{R}$, by the arithmetic-geometric mean inequality, we have

$$f\Big(\frac{x+z}{2}\Big) = a^{(x+z)/2} = \sqrt{a^{x+z}} = \sqrt{a^x a^z} \le \frac{a^x + a^z}{2} = \frac{f(x)+f(z)}{2}.$$

Since f is continuous, by Theorem 5.2.2 it is convex. By Theorem 5.2.1, f is differentiable at some (at many) points. As shown in the end of subsection 4.6, this implies that f is differentiable at all points of $\mathbb{R}$, with $f'(x) = cf(x)$, $x \in \mathbb{R}$, where $c = f'(0)$.

The inequality (5.3) for convex functions can be extended by induction to the case of n points; it is then called *Jensen's inequality*:

Theorem 5.2.3. *Let f be a convex function on an interval I and let $x_1, \dots, x_n \in I$; then for any $t_1, \dots, t_n > 0$ with $\sum_{i=1}^n t_i = 1$ we have $f\big(\sum_{i=1}^n t_i x_i\big) \le \sum_{i=1}^n t_i f(x_i)$.*

Proof. For $n = 2$ this is just (5.3). Given $t_1, \dots, t_{n+1} > 0$ with $\sum_{i=1}^{n+1} t_i = 1$, let $t = \sum_{i=1}^n t_i$ and $s_i = t_i/t$, $i = 1, \dots, n$, then $s_i > 0$ for all i and $\sum_{i=1}^n s_i = \big(\sum_{i=1}^n t_i\big)/t = 1$, so we may assume by induction that $f\big(\sum_{i=1}^n s_i x_i\big) \le \sum_{i=1}^n s_i f(x_i)$. Further, $t, t_{n+1} > 0$ and $t + t_{n+1} = 1$, so for $x = \sum_{i=1}^n s_i x_i$ we have

$$f\big(tx + t_{n+1}x_{n+1}\big) \le tf(x) + t_{n+1}f(x_{n+1}) \le t\sum_{i=1}^{n+1} s_i f(x_i) + t_{n+1}f(x_{n+1}) = \sum_{i=1}^{n+1} t_i f(x_i).$$

Since $tx + t_{n+1}x_{n+1} = \sum_{i=1}^{n+1} t_i x_i$, we have the induction step. ■

5.3. Natural exponential and logarithmic functions

Let $a > 0$, $\exp_a(x) = a^x$. We know that $\exp_a$ is diferentiable, let $\exp_a'(0) = c$. Put $b = a^{1/c}$, then $\exp_b(x) = b^x = (a^{1/c})^x = a^{x/c} = \exp_a(x/c)$, so $\exp_b'(0) = \exp_a'(0)/c = c/c = 1$. Since $\exp_b^{-1} = \log_b$ and $\exp_b(0) = 1$, we have that $\log_b'(1) = 1/\exp_b'(0) = 1$, that is, $\lim_{h\to 0} \frac{\log_b(1+h)}{h} = \lim_{h\to 0} \frac{\log_b(1+h) - \log_b 1}{h} = 1$. Since $\exp_b$ is continuous, this implies that $\lim_{h\to 0} \exp_b\big(\frac{\log_b(1+h)}{h}\big) = \exp_b(1) = b$. Since

$$\exp_b\big(\tfrac{\log_b(1+h)}{h}\big) = b^{(\log_b(1+h))/h} = \big(b^{\log_b(1+h)}\big)^{1/h} = (1+h)^{1/h},$$

we obtain that $b = \lim_{h\to 0}(1+h)^{1/h}$. But we know that for the sequence $h_n = 1/n$, which converes to 0, $\lim_{n\to\infty}(1+h_n)^{1/h_n} = \lim_{n\to\infty}(1+1/n)^n = e$, Euler's number. (We defined e this way!) So, our $b = e$. The exponential function $\exp_e(x) = e^x$ is denoted by just exp and called *the natural exponential function*; its inverse $\log_e$ is denoted by log (or by ln in some books) and called *the natural logarithm*. We have proved the following:

Theorem 5.3.1. (i) $\exp'(0) = 1$, *that is,* $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$. *As functions,* $(e^x)' = e^x$, $\exp' = \exp$.
(ii) $\log'(1) = 1$, *that is,* $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$. *As functions,* $\log'(x) = 1/x$.

As we will see below (see Theorem 5.5.6), the function exp is characterized by the property $\exp' = \exp$. The function exp is increasing and convex, with $\lim_{x\to-\infty} \exp x = 0$ and $\lim_{x\to+\infty} \exp x = +\infty$, so that $\exp(\mathbb{R}) = (0, +\infty)$. Respectively, log is increasing and concave, with $\log(0, +\infty) = \mathbb{R}$.

All exponential, logaritghmic, and power functions are expressible in terms of exp and log: for any $a > 0$, $\exp_a x = a^x = e^{x\log a}$; for any $a > 0$, $a \ne 1$, $\log_a x = \log x/\log a$; and for any $\alpha \in \mathbb{R}$, $x^\alpha = e^{\alpha \log x}$. We therefore have $\exp_a'(x) = \log a \exp_a x$, $\log_a'(x) = 1/(x\log a)$, and $(x^\alpha)' = x^\alpha \alpha/x = \alpha x^{\alpha-1}$.

5.4. Local properties of differentiable functions

Let $f\colon A \longrightarrow \mathbb{R}$ be a function and $a \in A$; we say that f is *increasing at* a if there is $\delta > 0$ such that $f(x) \le f(a)$ for all $x \in A \cap (a-\delta, a)$ and $f(x) \ge f(a)$ for all $x \in A \cap (a, a+\delta)$. We say that f is *strictly increasing at* a if there is $\delta > 0$ such that $f(x) < f(a)$ for all $x \in A \cap (a-\delta, a)$ and $f(x) > f(a)$ for all $x \in A \cap (a, a+\delta)$. We say that f is *decreasing at* a if there is $\delta > 0$ such that $f(x) \ge f(a)$ for all $x \in A \cap (a-\delta, a)$ and $f(x) \le f(a)$ for all $x \in A \cap (a, a+\delta)$, and that f is *strictly decreasing at* a if there is $\delta > 0$ such that $f(x) > f(a)$ for all $x \in A \cap (a-\delta, a)$ and $f(x) < f(a)$ for all $x \in A \cap (a, a+\delta)$.

Theorem 5.4.1. *Let f be a function differentiable at a point a. If f is increasing at a then $f'(a) \ge 0$; if f is decreasing at a then $f'(a) \le 0$. If $f'(a) > 0$ then f is strictly increasing at a; if $f'(a) < 0$ then f is strictly decreasing at a.*

Proof. If f is increasing at a, then $f(x) \ge f(a)$ so $f(x) - f(a) \ge 0$ for all $x > a$ in a neighborhood of a, so $\frac{f(x)-f(a)}{x-a} \ge 0$ for such x, so $f'(a) = \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} \ge 0$. If f is decreasing at a, then, similarly, $\frac{f(x)-f(a)}{x-a} \le 0$ for all $x > a$ in a neighborhood of a, so $f'(a) = \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} \ge 0$.

Let $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a) > 0$, then $\frac{f(x)-f(a)}{x-a} > 0$ for all x in a neighborhood I of a, that is, for $x \in I$, $f(x) - f(a) > 0$ if $x > a$ and $f(x) - f(a) < 0$ if $x < 0$. Similarly, if $f'(a) < 0$, then for all x in a neighborhood of a, $f(x) - f(a) < 0$ if $x > a$ and $f(x) - f(a) > 0$ if $x < 0$. ■

Let $f\colon A \longrightarrow \mathbb{R}$ be a function and $a \in A$; we say that f has *a local maximum at* a if there is $\delta > 0$ such that $f(a) \ge f(x)$ for all $x \in A \cap (a-\delta, a+\delta)$, *a strict local maximum at* a if there is $\delta > 0$ such that $f(a) > f(x)$ for all $x \in A \cap (a-\delta, a+\delta) \setminus \{a\}$, we say that f has *a local minimum at* a if there is $\delta > 0$ such that $f(a) \le f(x)$ for all $x \in A \cap (a-\delta, a+\delta)$, *a strict local minimum at* a if there is $\delta > 0$ such that $f(a) < f(x)$ for all $x \in A \cap (a-\delta, a+\delta) \setminus \{a\}$. If f has a local maximum or a local minimum at a, we say that f has *a local extremum at* a, or that a is *an extremal point of* f.

If a function f is defined in a neighborhood of a point a and has a local extremum at a, then f neither strictly increases nor strictly decreases at a. So, as a corollary, we obtain:

Theorem 5.4.2. *If a function f is differentiable at a point a and has a local extremum at a, then $f'(a) = 0$.*

The points x at which $f'(x) = 0$ are called *critical points* of f; Theorem 5.4.2 says that if x is an extremal point of f and f is differentiable at a, then x is a critical point of f. (Equivalently, if x is an extremal point of f then either x is a critical point of f or f is not differentiable at x.)

5.5. Mean value theorems

Recall that a function f is *differentiable on a set* A if f is differentiable at every point of A; this is the case, $f'(x)$, $x \in A$, is a function on A, called *the derivative* of f.

The following facts follow directly from the definition of the derivative:

Theorem 5.5.1. *Let f be a function differentiable on a set A. If f is constant, then $f' = 0$. If f is increasing, then $f' \ge 0$; if f is decreasing, then $f' \le 0$. If f is Lipschitz, $|f(x) - f(y)| \le C|x-y|$ for some $C \in \mathbb{R}$ for all $x, y \in A$, then $|f'| \le C$. If A is an interval and f convex or concave, then f' is increasing or, respectively, decreasing on A.*

For functions differentiable on an interval the converses of the assertions of Theorem 5.5.1 are also true, but they are difficult to prove directly. *Mean value theorems* are very handy for deducing properties of functions differentiable on intervals from properties of their derivatives.

The simplest one is *Rolle's theorem*:

Theorem 5.5.2. *Let f be a function continuous on a closed bounded interval $[a, b]$, differentiable on (a, b), and satisfying $f(a) = f(b)$. Then $f'(c) = 0$ for some $c \in (a, b)$.*

Proof. Since f is continuous on $[a, b]$, it attains its maximal value M and its minimal value m on $[a, b]$. If both values are taken at the endpoints a and b, then $M = f(a) = f(b) = m$, so f is constant on $[a, b]$, and $f'(c) = 0$ for all $c \in (a, b)$. Otherwise, at least one of the values M and m is taken at an interior point $c \in (a, b)$, so f has a global (so local) extremum at c, so $f'(c) = 0$ by Theorem 5.4.2. ■

The most applicable mean value theorem is *Lagrange's mean value theorem*, or just *the Mean Value Theorem, M.V.T.*:

Theorem 5.5.3. *Let f be a function continuous on a closed bounded interval $[a,b]$ and differentiable on (a,b). Then for some $c \in (a,b)$, $f(b)-f(a) = f'(c)(b-a)$.*

Proof. We will deduce Lagrange's theorem from Rolle's theorem. Consider the function $h(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$. Then h is continuous on $[a,b]$, differentiable on (a,b), $h(a) = f(a)$, and $h(b) = f(b) - \frac{f(b)-f(a)}{b-a}(b-a) = f(a)$. So, there is $c \in (a,b)$ such that $h'(c) = 0$. But $h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$, so $\frac{f(b)-f(a)}{b-a} = f'(c)$. ■

Lagrange's M.V.T. has a lot of important corollaries:

Theorem 5.5.4. *Let function f be differentiable on an interval I and $f'(x) = 0$ for all $x \in I$. Then f is constant on I.*

Proof. Let $x, y \in I$, $x < y$. Then there is $c \in (x,y)$ such that $f(y) - f(x) = f'(c)(y-x) = 0$, so $f(x) = f(y)$. ■

Theorem 5.5.5. *Suppose functions f and g are differentiable on an interval I and $f'(x) = g'(x)$ for all $x \in I$. Then $f = g + c$ for some $c \in \mathbb{R}$.*

Proof. $(f(x) - g(x))' = f'(x) - g'(x) = 0$ for all $x \in I$, so $f - g = \text{const}$. ■

As one more corollary we can now prove that the property $\exp' = \exp$ characterizes exp, up to scaling:

Theorem 5.5.6. *Let f be a function differentiable on $\mathbb{R}$ and satisfying $f' = f$. Then $f = c\exp$ for some $c \in \mathbb{R}$.*

Proof. $\left(\frac{f}{\exp}\right)' = \frac{f'\exp - f\exp'}{\exp^2} = \frac{f\exp - f\exp}{\exp^2} = 0$, so $f/\exp = \text{const}$. ■

Theorem 5.5.7. *Let function f be differentiable on an interval I. If $f'(x) \geq 0$ for all $x \in I$, then f is increasing on I. If $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I. If $f'(x) \leq 0$ for all $x \in I$, then f is decreasing on I. If $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing on I.*

Proof. Let $x, y \in I$, $x < y$; find $c \in (x,y)$ be such that $f(y) - f(x) = f'(c)(y-x)$. Then if $f'(c) \geq 0$ then $f(y) \geq f(x)$; if $f'(c) > 0$ then $f(y) > f(x)$; if $f'(c) \leq 0$ then $f(y) \leq f(x)$; if $f'(c) < 0$ then $f(y) < f(x)$. ■

Theorem 5.5.7 is used to investigate the behavior of a function differentiable on an interval: a function f strictly increases on the intervals where $f' > 0$ and strictly decreases on the intervals where $f' < 0$. Recall that the points x at which $f'(x) = 0$ are called *critical points* of f; we know that local extremum points of f are critical (assuming that d is differentiable at these points), but the converse is only true if f' "switches sign" at a:

Theorem 5.5.8. *Let f be differentiable in a neighbohood of a point a and $f'(a) = 0$. If for some $\delta > 0$, $f'(x) \geq 0$ for all $x \in (a-\delta, a)$ and $f'(x) \leq 0$ for all $x \in (a, a+\delta)$, then f has a local maximum at a; if $f'(x) > 0$ for all $x \in (a-\delta, a)$ and $f'(x) < 0$ for all $x \in (a, a+\delta)$, then f has a strict local maximum at a; if $f'(x) \leq 0$ for all $x \in (a-\delta, a)$ and $f'(x) \geq 0$ for all $x \in (a, a+\delta)$, then f has a local minimum at a; if $f'(x) < 0$ for all $x \in (a-\delta, a)$ and $f'(x) > 0$ for all $x \in (a, a+\delta)$, then f has a strict local minimum at a. If $f'(x) \geq 0$ for all $x \in (a-\delta, a+\delta)$, or $f'(x) \leq 0$ for all $x \in (a-\delta, a+\delta)$, then a is not a point of local extremum for f.*

Proof. Let $\delta > 0$ and $f'(x) \geq 0$ for all $x \in (a-\delta, a)$ and $f'(x) \leq 0$ for all $x \in (a, a+\delta)$. Then for any $x \in (a-\delta, a)$ there exists $z \in [x, a]$ such that $f(a) - f(x) = f'(z)(a-x) \geq 0$, so $f(x) \leq f(a)$; and for any $x \in (a, a+\delta)$ there exists $z \in [a, x]$ such that $f(x) - f(a) = f'(z)(x-a) \leq 0$ so $f(x) \leq f(a)$.

All other assertions are proved similarly. (The last one is a corollary of Theorem 5.5.7.) ■

Theorem 5.5.9. *Let function f be differentiable on an interval I and $f'(x)$ be bounded on I, $|f'(x)| \leq C$ for all $x \in I$. Then f is Lipschitz on I with constant C, $|f(x) - f(y)| \leq C|x-y|$ for al $x, y \in I$.*

Proof. Let $x, y \in I$, $x < y$; let $c \in (x, y)$ be such that $f(y) - f(x) = f'(c)(y - x)$, then $|f(y) - f(x)| = |f'(c)| \cdot |y - x| \le C|y - x|$. ■

Finally, here is a sort of converse of Theorem 5.2.1:

Theorem 5.5.10. *Let f be a function differentiable on an open interval I. If f' is an increasing function on I, then f is convex on I; if f' is strictly increasing on I, then f is strictly convex on I; if f' is decreasing on I, then f is concave on I; if f' is strictly decreasing on I, then f is strictly concave on I.*

Proof. Let $x, y, z \in I$, $x < y < z$. There are $c \in (x, y)$ and $d \in (y, z)$ such that $\frac{f(y)-f(x)}{y-x} = f'(c)$ and $\frac{f(z)-f(y)}{z-y} = f'(d)$. Since $c < y < d$, $f'(c) \le f'(d)$, so $\frac{f(y)-f(x)}{y-x} \le \frac{f(z)-f(y)}{z-y}$. Hence, f is convex. The other assertions are proved similarly. ■

Examples. Since $\exp' = \exp$ is an increasing function, exp is convex. Since $\log' x = 1/x$ is decreasing, log is concave.

Cauchy's mean value theorem is more general than Lagrange's M.V.T. (the latter corresponds to the case $g(x) = x$ of the former):

Theorem 5.5.11. *Let f and g be functions continuous on a closed bounded interval $[a, b]$ and differentiable on (a, b). Then there is $c \in (a, b)$ such that $\big(f(b) - f(a)\big)g'(c) = \big(g(b) - g(a)\big)f'(c)$; if $g(b) \ne g(a)$ and $g'(c) \ne 0$, this is equivalent to $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.*

Proof. Define the function $h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$. h is continuous on $[a, b]$ and differentiable on (a, b); also, $h(a) = f(b)g(a) - f(a)g(a) - g(b)f(a) + g(a)f(a) = f(b)g(a) - f(a)g(b)$ and $h(b) = f(b)g(b) - f(a)g(b) - g(b)f(b) + g(a)f(b) = f(b)g(a) - f(a)g(b)$, so $h(b) = h(a)$. By Rolle's theorem, there is $c \in (a, b)$ such that $h'(c) = 0$, which means that $(f(b) - f(a))g'(c) - (g(b) - g(a))f'(c) = 0$. ■

5.6. Discontinuities of the derivative functions

We will see later via the Fundamental theorem of calculus) that every continuous (on an interval) function f is a derivative of another function, called *a primitive* of f. Not every discontinuous function has a primitive, the derivative functions have some special properties.

The following theorem implies that the derivative function on an interval can only have discontinuities of the second kind:

Theorem 5.6.1. *Let function f be differentiable on an interval (a, b) and assume that a finite $\lim_{x\to a^+} f'(x)$ exists. Then a finite $\lim_{x\to a^+} f(x)$ exists and if we define $f(a) = \lim_{x\to a^+} f(x)$ then f becomes right-hand differentiable at a with $f'_+(a) = \lim_{x\to a^+} f'(x)$.*

(Of course, a similar fact holds for the left-hand derivative, and for the two-sided derivative.)

Proof. Since a finite $\lim_{x\to a^+} f'(x)$ exists, f' is bounded in a right-side neighborhood $(a, a + \delta)$ of a, so f is Lipschitz, and so, uniformly continuous in this neighborhood. Hence, $p = \lim_{x\to a^+} f(x)$ exists, and if we define $f(a) = p$ then f becomes continuous on $[a, b)$. Next, by the M.V.T., for any $x \in (a, b)$, $\frac{f(x)-f(a)}{x-a} = f'(c_x)$ for some $c_x \in (a, x)$. As $x \longrightarrow a^+$, $c_x \longrightarrow a^+$ as well and $c_x \ne a$ for all $x > a$. (In more details: c_x is a function of x satisfying $a < c_x < x$, so $\lim_{x\to a^+} c_x = a$.) Let $\lim_{x\to a^+} f'(x) = d$, then $\lim_{x\to a^+} f'(c_x) = d$ as well by the theorem on the limit of the composition, so $f'_+(a) = \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} = d$. ■

Darboux's theorem says that the derivative function on an interval, even if it discontinuous, has "the intermediate value" property.

Theorem 5.6.2. *Let f be defined on an interval $[a, b]$, differentiable on (a, b), and both $f'_+(a)$ and $f'_-(b)$ exist. Then for any c with $f'_+(a) < c < f'_-(b)$ or $f'_-(b) < c < f'_+(a)$ there exists $x_0 \in (a, b)$ such that $f'(x_0) = c$.*

Proof. W.l.o.g. assume that $f'_+(a) < c < f'_-(b)$. Let $d = \frac{f(b)-f(a)}{b-a}$; assume w.l.o.g. that $c \le d$. Define $\varphi(x) = \frac{f(x)-f(a)}{x-a}$ for $x \in (a, b]$ and $\varphi(a) = f'_+(a)$; then φ is continuous on $[a, b]$, with $\varphi(a) = f'_+(a) < c \le d = \varphi(b)$. By the I.V.T. there exists $x_1 \in (a, b]$ such that $\varphi(x_1) = c$. By the M.V.T., there exists $x_0 \in (a, x_1)$ such that $f'(x_0) = \frac{f(x_1)-f(a)}{x_1-a} = \varphi(x_1) = c$.

■

5.7. Trigonometric functions

I am now going to introduce *the trigonometric functions*: arcsin, sin and cos, though we don't yet have all necessary tools for this. *A primitive* of a function f is a differentiable function F such that $F' = f$; it doesn't necessarily exist, but we will prove (via integration) that every continuous function on an interval does have a primitive. If f is a function on an interval I that has a primitive, then this primitive is defined uniquely up to a constant: if $f = F' = G'$ then $F = G + \mathrm{const}$. It follows that for every $x_0 \in I$ and $c \in \mathbb{R}$ there is a unique primitive F of f such that $F(x_0) = c$.

Recall that a function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is said to be *even* if $f(-x) = f(x)$ for all x and *odd* if $f(-x) = -f(x)$ for all x. A function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is said to be *periodic with period* a, or a*-periodic*, for some $a > 0$, if $f(x + a) = f(x)$ for all $x \in \mathbb{R}$.

We define the function arcsin as a primitive of $f(x) = \frac{1}{\sqrt{1-x^2}}$: arcsin is a differentiable function on $(-1, 1)$ satisfying $\arcsin' = f$ and $\arcsin 0 = 0$. f is positive, decreasing on $(-1, 0)$ and increasing on $(0, 1)$, thus arcsin is strictly increasing on $(-1, 1)$, concave on $(-1, 0)$ and convex on $(0, 1)$. Since f is an even function and $\arcsin 0 = 0$, arcsin is odd. (Indeed, let $g(x) = \arcsin x + \arcsin(-x)$ for $x \in (-1, 1)$, then $g'(x) = f(x) - f(-x) = 0$ on $(-1, 1)$, so $g = \mathrm{const}$, so $g = g(0) = 2 \arcsin 0 = 0$.)

Since arcsin is increasing, $\lim_{x\to 1^-} \arcsin x$ exists; we will show that it is finite. For any $x \in (0, 1)$, $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}} > 1$ and $\arcsin'(x) = \frac{1}{\sqrt{1-x}\sqrt{1+x}} < \frac{1}{\sqrt{1-x}} = h'(x)$ for $h(x) = 2 - 2\sqrt{1-x}$ satisfying $h(0) = 0$; thus $x < \arcsin x < h(x)$ for all $x \in (0, 1)$, so $1 < \lim_{x\to 1^-} \arcsin x \le h(1) = 2$. The number $2 \lim_{x\to 1^-} \arcsin x$ is called *pi* and is denoted by π; we see that $2 < \pi \le 4$. Since arcsin is odd, we also have $\lim_{x\to -1^+} \arcsin x = -\frac{\pi}{2}$. We therefore can extend arcsin by continuity to the closed interval $[-1, 1]$ by $\arcsin(-1) = -\frac{\pi}{2}$ and $\arcsin 1 = \frac{\pi}{2}$.

Now, we define *the sine function* sin as the inverse of arcsin; it is an odd continuous strictly increasing function on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, with $\sin\left(-\frac{\pi}{2}\right) = -1$ and $\sin\left(\frac{\pi}{2}\right) = 1$. sin is differentiable on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, with $\sin' x = 1/\arcsin'(\sin x) = 1/(1/\sqrt{1 - \sin^2 x}) = \sqrt{1 - \sin^2 x}$. Since $\lim_{x\to -(\frac{\pi}{2})^+} \sin' x = \lim_{x\to (\frac{\pi}{2})^-} \sin' x = 0$, by Theorem 5.6.1, $\sin'_+\left(-\frac{\pi}{2}\right)$ and $\sin'_-\left(\frac{\pi}{2}\right)$ also exist and $= 0$.

Next, we extend sin as follows: define $\sin x = \sin(\pi - x)$ for all $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, and then extend it to whole $\mathbb{R}$ *by* 2π*-periodicity*: for every $x \in \mathbb{R}$ find $n \in \mathbb{Z}$ such that $x - 2n\pi \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ (such n is unique, it is the integer part of $\left(x + \frac{\pi}{2}\right)/(2\pi)$) and define $\sin x = \sin(x - 2n\pi)$. Then sin is continuous and differentiable everywhere except, perhaps, the "glueing points" $\frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$; but at these points sin is also continuous and differentiable since $\sin\left(\frac{\pi}{2} + 2n\pi\right) = 1$, $\sin\left(-\frac{\pi}{2} + 2n\pi\right) = 1$, and $\sin'_+\left(\frac{\pi}{2} + n\pi\right) = \sin'_-\left(\frac{\pi}{2} + n\pi\right) = 0$ for all $n \in \mathbb{Z}$. By definition, sin is periodic with period 2π.

The function $\cos = \sin'$ is called *cosine*; it is 2π-periodic and even. We have $\cos x = \sin' x = \sqrt{1 - \sin^2 x}$ for all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; in particular, $\cos(0) = 1$, and $\cos\left(-\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$. On this interval $\cos' x = \left(\sqrt{1 - \sin^2 x}\right)' = \frac{1}{2\sqrt{1-\sin^2 x}}(-2 \sin x \cos x) = -\sin x$. On $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, $\cos x = \sin' x = -\sin'(\pi - x) = -\sqrt{1 - \sin^2(\pi - x)} = -\sqrt{1 - \sin^2 x}$, so also $\cos' x = \left(-\sqrt{1 - \sin^2 x}\right)' = \frac{-1}{2\sqrt{1-\sin^2 x}}(-2 \sin x \cos x) = -\sin x$. So, we obtain that $\cos' = -\sin$, and that $\cos x = \pm\sqrt{1 - \sin^2 x}$ so $\cos^2 x + \sin^2 x = 1$ for all $x \in \mathbb{R}$.

We are now going to obtain "the addition formulas" for sin and cos. We first show that the property $\sin' = \cos$ and $\cos' = -\sin$ characterizes "linear combinations" of sin and cos:

Theorem 5.7.1. *Let f and g be functions differentiable on $\mathbb{R}$ and satisfying $f' = g$ and $g' = -f$. Then $f = a \cos x + b \sin x$ and $g = -a \sin x + b \cos x$, where $a = f(0)$ and $b = g(0)$.*

Proof. We first prove that if $f(0) = g(0) = 0$ then $f = g = 0$. We have $(f^2 + g^2)' = 2ff' + 2gg' = 2fg - 2gf = 0$, so $f^2 + g^2 = \mathrm{const} = f(0)^2 + g(0)^2 = 0$. But this is only possible if $f = g = 0$.

For the general case, define $\widetilde{f} = f - f(0) \cos - g(0) \sin$ and $\widetilde{g} = g + f(0) \sin - g(0) \cos$. Then $\widetilde{f}$ and $\widetilde{g}$ also satisfy

$$\widetilde{f}' = f' + f(0) \cos - g(0) \sin = \widetilde{g}, \quad \widetilde{g}' = g' + f(0) \cos + g(0) \sin = -\widetilde{f},$$

with $\widetilde{f}(0) = f(0) - f(0) = 0$ and $\widetilde{g}(0) = g(0) - g(0) = 0$. So, $\widetilde{f} = \widetilde{g} = 0$, thus $f = f(0)\cos + g(0)\sin$ and $g = -f(0)\sin + g(0)\cos$. ■

Now let $y \in \mathbb{R}$, and consider the functions $f(x) = \sin(x+y)$ and $g(x) = \cos(x+y)$. We have $f' = g$, $g' = f$, $f(0) = \sin y$ and $g(0) = \cos y$, so $f(x) = \sin y \cos x + \cos y \sin x$ and $g(x) = -\sin y \sin x + \cos y \cos x$. In particular, $\sin\left(\frac{\pi}{2} - x\right) = \sin(-x)\cos\left(\frac{\pi}{2}\right) + \cos(-x)\sin\left(\frac{\pi}{2}\right) = \cos x$ and $\cos\left(\frac{\pi}{2} - x\right) = \cos\left(x - \frac{\pi}{2}\right) = \sin x$.

Let's summarize:

Theorem 5.7.2. $\sin$ *and* $\cos$ *are differentiable functions* $\mathbb{R} \longrightarrow \mathbb{R}$, *satisfying* $\sin' = \cos$ *and* $\cos' = -\sin$. $\sin$ *is odd,* $\cos$ *is even. Both* $\sin$ *and* $\cos$ *are periodic with period* 2π. $\sin^2 + \cos^2 = 1$. $\mathrm{Rng}(\sin) = [-1, 1]$, $\sin$ *increases from* -1 *to* 1 *on the intervals of the form* $\left[-\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi\right]$ *and decreases from* 1 *to* -1 *on the intervals of the form* $\left[\frac{\pi}{2} + 2n\pi, \frac{3\pi}{2} + 2n\pi\right]$, $n \in \mathbb{Z}$; $\sin x = 0$ *iff* $x = n\pi$ *for some* $n \in \mathbb{Z}$; $\sin x = 1$ *iff* $x = \frac{\pi}{2} + 2n\pi$ *for some* $n \in \mathbb{Z}$; $\sin x = -1$ *iff* $x = -\frac{\pi}{2} + 2n\pi$ *for some* $n \in \mathbb{Z}$. $\sin$ *is concave on the intervals of the form* $[2n\pi, \pi + 2n\pi]$ *and convex on the intervals of the form* $[-\pi + 2n\pi, 2n\pi]$, $n \in \mathbb{Z}$. *For any* x *and* y, $\sin(x+y) = \sin x \cos y + \cos x \sin y$ *and* $\cos(x+y) = \cos x \cos y - \sin x \sin y$. *It follows that for any* x, $\sin(x+\pi) = -\sin x$ *and that* $\cos x = \sin\left(x + \frac{\pi}{2}\right)$; *thus* $\cos$ *is obtained from* $\sin$ *by "shifting" it by* $\frac{\pi}{2}$ *(and thus whatever is true for* $\sin$ *is true for* $\cos$ *up to this shift).*

The functions $\sin x$ and $\sin(1/x)$ are very useful for constructing examples of functions with "prescribed singularities":

$\lim_{x\to\infty} \sin x$ doesn't exist.

For $f(x) = x^{-1}\sin x^2 = 0$, $\lim_{x\to\infty} f(x)$ exists, but $f'(x) = 2\cos x^2 - x^{-2}\sin x^2$ has no limit as $x \longrightarrow \infty$.

The function $f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is differentiable on $\mathbb{R} \setminus \{0\}$, and has no limit (and so, has a second-kind discontinuity) at 0: we have $f\left(\frac{1}{n\pi}\right) = 0$ for all $n \in \mathbb{N}$, $f\left(\frac{1}{\pi/2 + 2n\pi}\right) = 1$ for all $n \in \mathbb{N}$, and $f\left(\frac{1}{-\pi/2 + 2n\pi}\right) = -1$ for all $n \in \mathbb{N}$, while all the sequences $\left(\frac{1}{n\pi}\right)$, $\left(\frac{1}{\pi/2+2n\pi}\right)$, $\left(\frac{1}{-\pi/2+2n\pi}\right)$ tend to 0.

The function $f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, is differentiable on $\mathbb{R} \setminus \{0\}$, is Lipschitz (and so continuous) at 0, but not differentiable at 0.

The function $f(x) = \begin{cases} \sqrt{x}\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, is differentiable on $\mathbb{R} \setminus \{0\}$, is continuous but not Lipschitz at 0 (and so, is not differentiable at 0).

The function $f(x) = \begin{cases} x^2\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, is differentiable on $\mathbb{R}$, with $f'(0) = \lim_{x\to 0} \frac{x^2\sin(1/x)}{x} = 0$, but $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$ is discontinuous at 0.

The function $f(x) = \begin{cases} x^2\sin\frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, is differentiable on $\mathbb{R}$, with $f'(0) = 0$, but $f'(x) = 2x\sin\frac{1}{x^2} - 2\frac{1}{x}\cos\frac{1}{x^2}$ is discontinuous at 0 and unbounded in any neighborhood of 0. Thus, f is Lipschitz at every point, but is not Lipschitz in any neighborhood of 0.

The functions $\sin$, $\cos$, $\tan = \frac{\sin}{\cos}$, $\cot = \frac{\cos}{\sin}$, $\sec = \frac{1}{\cos}$, $\mathrm{cosec} = \frac{1}{\sin}$ are called *trigonometric*. The functions $\arcsin$, $\arccos = \cos^{-1}$, $\arctan = \tan^{-1}$, and $\mathrm{arccot} = \cot^{-1}$ are called *inverse trigonometric functions*.

Functions, obtainable by adding, multiplying, dividing, and taking compositions of the constant, power, exponential, logarithmic, trigonometric functions and their inverses are called *elementary functions*. Elementary functions are differentiable on the intervals on which they are defined, and their derivatives are also elementary.

5.8. Higher order derivatives

If the derivative function f' of a function f is itself differentiable at a point a (which assume that f' is defined in a neighborhood of a) the derivative $(f')'(a)$ is called *the second derivative* of f at a and is denoted by $f''(a)$ or $f^{(2)}(a)$. If f has the second derivative at all points of a set A, the function $f''(x) = f^{(2)}(x)$, $x \in A$, is called the second derivative function of f on A. By induction, for any $n \in \mathbb{N}$, if the n-th derivative $f^{(n)}$ of f is (defined in a neighborhood of a and) is differentiable at a, *the* $(n+1)$*-st derivative of* f at a

is $f^{(n+1)}(a) = (f^{(n)})'(a)$, and if $f^{(n+1)}(x)$ is defined for all $x \in A$, this function is called the $(n+1)$-st derivative of f on A. (It is sometimes convenient to denote f itself by $f^{(0)}$ and f' by $f^{(1)}$.)

By induction on n, if two functions f and g are n-times differentiable at a point a (or on a set A), then $f+g$, fg, f/g (if $g(a) \neq 0$) are also differentiable at a (respectively on A), and $(f+g)^{(n)}(a) = f^{(n)}(a) + g^{(n)}(a)$, $(fg)^{(n)}(a) = \sum_{i=0}^{n} \binom{n}{i} f^{(n-i)}(a) g^{(i)}(a)$. (The formula for $(f/g)^{(n)}(a)$ is not simple.) If f is n-times differentiable at a point a (or on a set A) and g is n-times differentiable at $f(a)$ (respectively, on the set $f(A)$), then $g \circ f$ is n-times differentiable at a (respectively, on A). If f is invertible, n-times differentiable at a (or on A) continuous in a neighborhood of a (of A), and $f'(a) \neq 0$ (respectively, on A), then f^{-1} is n-times differentiable at $f(a)$ (respectively, on A).

A function f is said to be *infinitely differentiable* on a set A if f is n-times differentiable on A for all $n \in \mathbb{N}$. Functions, infinitely differentiable on their domain, are also called *smooth*. All elementary functions are smooth.

The set of functions, continuous on a set A, is denoted by $C(A)$. (So, "$f \in C(A)$" means that f is continuous on A.) The set of functions f which are *continuously differentiable on* A, that is, such that f is n-times differentiable on A and $f^{(n)}$ is continuous on A, is denoted by $C^n(A)$. The set of functions infinitely differentiable on A is denoted by $C^\infty(A)$.

Here are some applications of the second derivative, a local:

Theorem 5.8.1. *Let function f be twice differentiable at a with $f'(a) = 0$. If $f''(a) > 0$ then f has a local minimum at a, if $f''(a) < 0$ then f has a local maximum at a.*

Proof. If $f''(a) > 0$ then f' is increasing at a (by Theorem 5.4.1), that is, $f'(x) < 0$ for all x in a left-hand neighborhood $(a-\delta, a)$ of a and $f'(x) > 0$ for all x in a right-hand neighborhood $(a, a+\delta)$ of a, so f has a local minimum at a (by Theorem 5.5.8). The proof of the second assertion is similar. ∎

and a global:

Theorem 5.8.2. *Let function f be twice-differentiable on an interval I. Then f is convex on I iff $f'' \geq 0$ on I and f is concave on I iff $f'' \leq 0$ on I. If $f'' > 0$ on I then f is strictly convex on I and if $f'' < 0$ on I then f is strictly concave on I.*

Proof. f is convex on I iff f' is increasing on I, and f' is increasing on I iff $f'' \geq 0$ on I. All other assertions are proved similarly. ∎

Example. Since $(x^\alpha)'' = \alpha(\alpha-1)x^{\alpha-2}$, the function x^α is convex on $(0, +\infty)$ if $\alpha \geq 1$ or ≤ 0 and is concave if $0 \leq \alpha \leq 1$.

We know that $f' = 0$ iff f is constant; here is a generalization of this fact:

Theorem 5.8.3. *Let function f be n-times differentiable on an interval I. Then $f^{(n)} = 0$ on I iff f is a polynomial of degree $\leq (n-1)$.*

Proof. If f is a polynomial of degree $\leq (n-1)$, then f' is a polynomial of degree $\leq (n-2)$; so by induction, $f^{(n)} = (f')^{(n-1)} = 0$.

Conversely, let $f^{(n)} = 0$, then $(f')^{(n-1)} = 0$. By induction, f' is a polynomial of degree $\leq (n-2)$, $f'(x) = a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$. Define $g(x) = \frac{a_{n-2}}{n-1}x^{n-1} + \cdots + \frac{a_1}{2}x^2 + a_0 x$, then $g' = f'$, so $f = g + c$ for some $c \in \mathbb{R}$, so f is a polynomial of degree $\leq (n-1)$. ∎

5.9. L'Hospital's rule

L'Hospital's rule is a very handy tool for finding indeterminate limits of the form $0/0$ or ∞/∞; in short, it says that $\lim(f/g) = \lim(f'/g')$. However, its application requires some caution: the theorem is, actually, not that easy, and all its necessary conditions must be checked before its usage! Let's define *a punctured neighborhood* of $a \in \mathbb{R}$ as a set $(a-\delta, a+\delta) \setminus \{a\}$ with $\delta > 0$; of a^- as an interval $(a-\delta, a)$; of a^+ as an interval $(a, a+\delta)$; of $+\infty$ as an interval $(M, +\infty)$ for some $M > 0$; of $-\infty$ as an interval $(-\infty, -M)$; and of ∞ as the set $(-\infty, -M) \cup (M, +\infty)$.

Theorem 5.9.1. *Let α be any of a, a^-, a^+, where $a \in \mathbb{R}$, $+\infty$, $-\infty$, or ∞. Let f and g be functions defined and differentiable in a punctured neighborhood I of α and satisfying the following conditions:*
(i) $\lim_{x\to\alpha} f(x) = \lim_{x\to\alpha} g(x) = 0$ *or* $\lim_{x\to\alpha} f(x) = \lim_{x\to\alpha} g(x) = \infty$*;*

(ii) $g'(x) \neq 0$ *for all* $x \in I$*;*
(iii) $\lim_{x\to\alpha} f'(x)/g'(x)$ *exists, finite or infinite.*
Then $\lim_{x\to\alpha} f(x)/g(x)$ *also exists and equals* $\lim_{x\to\alpha} f'(x)/g'(x)$*.*

Proof. The theorem is, actually, a bunch of theorems in one box. Let's start with the case of $\alpha = a^+$, $I = (a, a+\delta)$ for some $\delta > 0$.

First, let $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$. Put $f(a) = g(a) = 0$, then f and g become right continuous at a. Since $g'(x) \neq 0$ for all $x \in I$, $g(x) \neq 0$ for all $x \in I$ by Rolle's theorem. Thus for any $x \in I$, by Cauchy's M.V.T., $\frac{f(x)}{g(x)} = \frac{f(x)-f(0)}{g(x)-g(0)} = \frac{f'(c_x)}{g'(c_x)}$ for some $c_x \in (a,x)$. As $x \longrightarrow a^+$ we have $c_x \longrightarrow a^+$ (c_x is a function of x with $a < c_x < x$ for all x), so $\lim_{x\to a^+} \frac{f'(c_x)}{g'(c_x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$. Hence, $\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$.

Now let $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = +\infty$. Let $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = b \in \mathbb{R}$. Let $\varepsilon > 0$; find $\delta_1 > 0$ such that $\delta_1 < \delta$ and $\left|\frac{f'(x)}{g'(x)} - b\right| < \varepsilon/2$ for all $x \in (a, a+\delta_1)$. Fix any $y \in (a, a+\delta_1)$. For any $x \in (a,y)$, since $g'(z) \neq 0$ for all $z \in (x,y)$ we have $g(x) \neq g(y)$ by Rolle's theorem, therefore by Cauchy's M.V.T., $\frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f'(c_x)}{g'(c_x)}$ for some $c_x \in (x,y)$; so $\left|\frac{f(y)-f(x)}{g(y)-g(x)} - b\right| < \varepsilon/2$. For any $x \in (a,y)$ we have $\frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f(x)}{g(x)} \cdot \frac{f(y)-f(x)}{f(x)} \cdot \frac{g(x)}{g(y)-g(x)}$. Since $f(x), g(x) \longrightarrow +\infty$ as $x \to a^+$, $\frac{f(x)}{f(y)-f(x)} \cdot \frac{g(y)-g(x)}{g(x)} \longrightarrow 1$; as $\frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f'(c_x)}{g'(c_x)}$ is bounded, this implies that $\left|\frac{f(y)-f(x)}{g(y)-g(x)} - \frac{f(x)}{g(x)}\right| = \left|\frac{f(y)-f(x)}{g(y)-g(x)}\right| \cdot \left|1 - \frac{f(x)}{f(y)-f(x)} \cdot \frac{g(y)-g(x)}{g(x)}\right| \longrightarrow 0$, so there is $\delta_2 > 0$, with $\delta_2 < y - a < \delta_1$, such that for all $x \in (a, a+\delta_2)$, $\left|\frac{f(y)-f(x)}{g(y)-g(x)} - \frac{f(x)}{g(x)}\right| < \varepsilon/2$. Then for all such x, $\left|\frac{f(x)}{g(x)} - b\right| < \varepsilon$. Hence, $\lim_{x\to a^+} \frac{f(x)}{g(x)} = b$.

Let, again, $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = +\infty$, and let now $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = +\infty$. Let $M > 0$, find $\delta_1 > 0$ with $\delta_1 < \delta$ such that $\frac{f'(x)}{g'(x)} > 2M$ for all $x \in (a, a+\delta_1)$. Fix any $y \in (a, a+\delta_1)$. Then for any $x \in (a,y)$, $g(x) \neq g(y)$ by Rolle's theorem, so by Cauchy's M.V.T., $\frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f'(c_x)}{g'(c_x)}$ for some $c_x \in (x,y)$, so $\frac{f(y)-f(x)}{g(y)-g(x)} > 2M$. Since $\frac{f(x)}{f(y)-f(x)} \cdot \frac{g(y)-g(x)}{g(x)} \longrightarrow 1$ as $x \to a^+$, there is $\delta_2 > 0$, with $\delta_2 < y - a < \delta_1$, such that for all $x \in (a, a+\delta_2)$, $\frac{f(x)}{f(y)-f(x)} \cdot \frac{g(y)-g(x)}{g(x)} > 1/2$, so $\frac{f(x)}{g(x)} = \frac{f(y)-f(x)}{g(y)-g(x)} \cdot \frac{f(x)}{f(y)-f(x)} \cdot \frac{g(y)-g(x)}{g(x)} > 2M/2 = M$. Hence, $\lim_{x\to a^+} \frac{f(x)}{g(x)} = +\infty$.

The case of $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = -\infty$ is obtained by replacing f by $-f$. If $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = \infty$, then since $g'(x) \neq 0$ in I, g' has constant sign on I by Darboux's theorem, and since $f'/g' \longrightarrow \infty$ as $x \to a^+$, $f'(x) \neq 0$ for all x in a right-side neighborhood of a as well, so f' also has constant sign; hence either $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = +\infty$ or $-\infty$.

For the other cases of α:
When $\alpha = a^-$ we can replace f and g by $f(-x)$ and $g(-x)$ respectively to reduce the situation to the case of $\alpha = -a^+$.
The case of $\alpha = a$ is a combination of two cases, of $\alpha = a^-$ and of $\alpha = a^+$.
The case of $\alpha = +\infty$ follows from the case of $\alpha = 0^+$ by replacing x by $1/x$: consider the functions $\widetilde{f}(x) = f(1/x)$ and $\widetilde{g}(x) = g(1/x)$, then $\widetilde{f}'(x) = -f'(1/x)/x^2$ and $\widetilde{g}'(x) = -g'(1/x)/x^2$; The functions $\widetilde{f}$ and $\widetilde{g}$ are defined in a right-side neighbohood of 0, with $\widetilde{g}'(x) \neq 0$ for all x in this neighborhood, satisfy $\lim_{x\to 0^+} \widetilde{f}(x) = \lim_{x\to+\infty} f(x) = \lim_{x\to 0^+} \widetilde{g}(x) = \lim_{x\to+\infty} g(x) = 0$ or ∞, and $\lim_{x\to 0^+} \frac{\widetilde{f}'(x)}{\widetilde{g}'(x)} = \lim_{x\to 0^+} \frac{-f'(1/x)/x^2}{-g'(1/x)/x^2} = \lim_{x\to+\infty} \frac{f'(x)}{g'(x)}$. So, $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{\widetilde{f}(x)}{\widetilde{g}(x)} = \lim_{x\to 0^+} \frac{\widetilde{f}'(x)}{\widetilde{g}'(x)} = \lim_{x\to+\infty} \frac{f'(x)}{g'(x)}$.
The case of $\alpha = -\infty$ is similar, the case of $\alpha = \infty$ is a combination of the cases of $\alpha = +\infty$ and $\alpha = -\infty$. ■

Examples. (i) $\lim_{x\to+\infty} \frac{x}{e^x} = \lim_{x\to+\infty} \frac{1}{e^x} = 0$. Thus, also $\lim_{x\to+\infty} \frac{x^2}{e^x} = \lim_{x\to+\infty} \frac{2x}{e^x} = 0$, and by induction, $\lim_{x\to+\infty} \frac{x^n}{e^x} = 0$ for all $n \in \mathbb{N}$.
(ii) $\lim_{x\to+\infty} \frac{\log x}{x} = \lim_{x\to+\infty} \frac{1/x}{1} = 0$. More generally, for any $\alpha > 0$, $\lim_{x\to+\infty} \frac{\log x}{x^\alpha} = \lim_{x\to+\infty} \frac{1/x}{\alpha x^{\alpha-1}} = \lim_{x\to+\infty} \frac{1}{\alpha x^\alpha} = 0$.
(iii) $\lim_{x\to 0^+} x\log x = \lim_{x\to 0^+} \frac{\log x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = 0$.
(iv) $\lim_{x\to 0} \frac{e^x-1-x}{x^2} = \lim_{x\to 0} \frac{e^x-1}{2x} = \frac{1}{2}$ since $\lim_{x\to 0} \frac{e^x-1}{x} = (e^x)'|_{x=0} = 1$.

(v) $\lim_{x\to 0}\frac{\sin x}{x}=\cos 0=1$ (since $\sin' 0=\cos 0$, no L'Hospital is needed), and $\lim_{x\to\infty}\frac{\cos x-1}{x}=-\sin 0=0$ (since $\cos' 0=-\sin 0$). By L'Hospital, $\lim_{x\to 0}\frac{\cos x-1}{x^2}=\lim_{x\to 0}\frac{-\sin x}{2x}=\frac{-1}{2}$.
(vi) $\infty=\lim_{x\to 0}\frac{\cos x}{x}\neq\lim_{x\to 0}\frac{-\sin x}{1}=0$, – since $\lim_{x\to 0}\cos x=1\neq 0$, L'Hospital's rule is not applicable.
(vii) $0=\lim_{x\to\infty}\frac{x+\sin x}{x}\neq\lim_{x\to\infty}\frac{1+\cos x}{1}$, the second limit doesn't even exist.
(viii) Here is *Stolz's "counterexample"* to L'Hospital's rule: Let $f(x)=x+\sin x\cos x$ and $g(x)=f(x)e^{\sin x}$, $x\in\mathbb{R}$, then $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}g(x)=\infty$ (since $\sin x\cos x$ is bounded and $e^{\sin x}$ oscillates between $1/e$ and e). Then $f'(x)=1+\cos^2 x-\sin^2 x=2\cos^2 x$, $g'(x)=(2\cos^2 x)e^{\sin x}+f(x)e^{\sin x}\cos x$, so $\frac{f'(x)}{g'(x)}=\frac{2\cos x}{2\cos x+f(x)}e^{-\sin x}\longrightarrow 0$ as $x\longrightarrow\infty$. However, $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}e^{-\sin x}$ doesn't exist! The reason why L'Hospital's rule fails is that the condition "$g'(x)\neq 0$ for all x in a neighborhood of ∞" is not satisfied.

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6. Riemann integral

6.1. Integrable functions

The integral $\int_I f$ of a nonnegative function f on an interval I is "the area of the region under the graph of f". There are different ways to define what this "area" is; we will study the easiest, *Riemann integral*. The disadvatage of Riemann integral is that it only applies to bounded *almost everywhere* continuous functions on intervals in $\mathbb{R}$ (or, at most, multidimensional intervals in $\mathbb{R}^d$).

Let $[a,b]$ be a (closed bounded) interval. *A partition* of $[a,b]$ is a finite set $P=\{x_0,x_1,\ldots,x_n\}$ such that $a=x_0<x_1<\ldots<x_n=b$. For each $i=1,\ldots,n$ let $\Delta x_i=x_i-x_{i-1}$. *The mesh* of P is $\operatorname{mesh}P=\max\{\Delta x_1,\ldots,\Delta x_n\}$.

Let $f\colon[a,b]\longrightarrow\mathbb{R}$ be a bounded function and P be a partition of $[a,b]$. For each $i=1,\ldots,n$, let $M_i=\sup\{f(x)\mid x\in[x_{i-1},x_i]\}$ and $m_i=\inf\{f(x)\mid x\in[x_{i-1},x_i]\}$; *the upper sum* of f with respect to P is $U(f,P)=\sum_{i=1}^n M_i\Delta x_i$, and *the lower sum* of f with respect to P is $L(f,P)=\sum_{i=1}^n m_i\Delta x_i$. Clearly, $L(f,P)\leq U(f,P)$.

A selection subordinate to P is a finite set $\sigma=\{z_1,\ldots,z_n\}$ such that $z_i\in[x_{i-1},x_i]$ for $i=1,\ldots,n$. *The Riemann sum* of f associated with P and such a selection σ is $S(f,P,\sigma)=\sum_{i=1}^n f(z_i)\Delta x_i$. Clearly, $L(f,P)\leq S(f,P,\sigma)\leq U(f,P)$.

For two partitions P, and P' of $[a,b]$, P' is said to be *a refinement* of P if $P\subseteq P'$.

Lemma 6.1.1. *If P' is a refinement of a partiton $P=\{x_0,x_1,\ldots,x_n\}$, then $U(f,P')\leq U(f,P)$ and $L(f,P')\geq L(f,P)$.*

Proof. P' is obtained from P by adding finitely many points, $P'=P\cup\{y_1,\ldots,y_m\}$; it suffices to prove the assertion for the case $P'=P\cup\{y\}$ and use induction. Let k be such that $y\in(x_{k-1},x_k)$. For all i, let $M_i=\sup\{f(x)\mid x\in[x_{i-1},x_i]\}$, and let $M_k'=\sup\{f(x)\mid x\in[x_{k-1},y]\}$ and $M_k''=\sup\{f(x)\mid x\in[y,x_k]\}$. Then $U(f,P)=\sum_{i=1}^n M_i\Delta x_i=\sum_{i\neq k}M_i\Delta x_i+M_k(x_k-x_{k-1})$ and $U(f,P')=\sum_{i\neq k}M_i\Delta x_i+M_k'(y-x_{k-1})+M_k''(x_k-y)$. But $M_k',M_k''\leq M_k$, so $M_k'(y-x_{k-1})+M_k''(x_k-y)\leq M_k(x_k-x_{k-1})$. So, $U(f,P')\leq U(f,P)$.

The proof of $L(f,P')\geq L(f,P)$ is similar. ∎

It follows that

Theorem 6.1.2. *For any two partitions P and Q of $[a,b]$, $L(f,P)\leq U(f,Q)$.*

Proof. $P\cup Q$ is a refinement of both P and Q, so $L(f,P)\leq L(f,P\cup Q)\leq U(f,P\cup Q)\leq U(f,Q)$. ∎

The number $U(f)=\inf\{U(f,P),\ P\text{ is a partition of }[a,b]\}$ is called *the upper integral* of f over $[a,b]$; $L(f)=\sup\{L(f,P),\ P\text{ is a partition of }[a,b]\}$ is called *the lower integral* of f over $[a,b]$. By Theorem 6.1.2, $L(f)\leq U(f)$; if $L(f)=U(f)$, f is said to be *integrable* on $[a,b]$, $L(f)=U(f)$ is called *the integral* of f over $[a,b]$ and is denoted by $\int f$, $\int_a^b f$, $\int_{[a,b]}f$, or $\int_a^b f(x)\,dx$.

Examples. (i) Dirichlet's function $f(x)=\begin{cases}1, & x\in\mathbb{Q}\\ 0, & x\notin\mathbb{Q}\end{cases}$ is non-integrable: for this function $U(f,P)=b-a$ and $L(f,P)=0$ for any partition P of $[a,b]$.

(ii) Any constant function $f = c$ is integrable, with $\int_a^b f = c(b-a)$: for any partition P, for any i we have $M_i = m_i = c$, so $U(f,P) = L(f,P) = \sum_{i=1}^n c\Delta x_i = c\sum_{i=1}^n \Delta x_i = c(b-a)$.

For a partition $P = \{x_0, x_1, \ldots, x_n\}$, let's define $\Delta(f,P) = U(f,P) - L(f,P)$. For a subinterval I of $[a,b]$, let $\operatorname{Var}_I f = \sup\{f(x),\ x \in I\} - \inf\{f(x),\ x \in I\} = \sup\{|f(x) - f(y)|,\ x,y \in I\}$. Then $\operatorname{Var}_{[x_{i-1},x_i]} f = M_i - m_i$ for all i, so $\Delta(f,P) = \sum_{i=1}^n (M_i - m_i)\Delta x_i = \sum_{i=1}^n \operatorname{Var}_{[x_{i-1},x_i]} f \Delta x_i$.

It follows from Lemma 6.1.1 that if P' is a refinement of P, then $\Delta(f,P') \leq \Delta(f,P)$.

Theorem 6.1.3. *A bounded function f is integrable on $[a,b]$ iff for any $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that $\Delta(f,P) < \varepsilon$. For such P we have $0 \leq U(f,P) - \int_a^b f < \varepsilon$, $0 \leq \int_a^b f - L(f,P) < \varepsilon$, and $\left|S(f,P,\sigma) - \int_a^b f\right| < \varepsilon$ for any selection σ subordinate to P.*

Proof. If for some $\varepsilon > 0$ a partition P is such that $U(f,P) - L(f,P) < \varepsilon$, then $U(f) - L(f) \leq U(f,P) - L(f,P) < \varepsilon$. If such a P exists for every $\varepsilon > 0$, then $U(f) - L(f) < \varepsilon$ for al $\varepsilon > 0$, so f is integrable.

Now suppose that f is integrable, and let $\varepsilon > 0$. By definition of $U(f)$ and $L(f)$, there are partitions Q and R of $[a,b]$ such that $U(f) \leq U(f,Q) < U(f) + \varepsilon/2$ and $L(f) - \varepsilon/2 < L(f,R) \leq L(f)$; then $U(f,Q) - L(f,R) < \varepsilon$. Put $P = Q \cup R$, then $L(f,R) \leq L(f,P) \leq U(f,P) \leq U(f,Q)$, so $\Delta(f,P) = U(f,P) - L(f,P) < \varepsilon$.

Since $L(f,P) \leq \int_a^b f \leq U(f,P)$ and $L(f,P) \leq S(f,P,\sigma) \leq U(f,P)$, we also have $0 \leq U(f,P) - \int_a^b f < \varepsilon$, $0 < \int_a^b f - L(f,P) < \varepsilon$, and $\left|S(f,P,\sigma) - \int_a^b f\right| < \varepsilon$. ■

Actually, if a function f is integrable, then any partition P with small enough mesh has a small $\Delta(f,P)$:

Theorem 6.1.4. *If f is integrable on $[a,b]$ then for any $\varepsilon > 0$ there is $\delta > 0$ such that for any partition P of $[a,b]$ with* $\operatorname{mesh} P < \delta$ *we have $\Delta(f,P) < \varepsilon$.*

Proof. Let $\varepsilon > 0$, let Q be a partition for which $\Delta(f,Q) < \varepsilon/2$. Let $|Q| = n$, let $|f| \leq M$; put $\delta = \varepsilon/(4Mn)$. Let P be a partition with $\operatorname{mesh} P < \delta$. Consider the partition $P' = P \cup Q$; since P' is a refinement of Q, we have $\Delta(f,P') < \varepsilon/2$. P' is obtained from P by adding at most n points, that is, by subdividing at most n intervals into two or more subintervals, and $\Delta(f,P')$ can be smaller than $\Delta(f,P)$ only because of them; the total contribution of these intervals into $\Delta(f,P)$ cannot be larger than $2Mn\delta = \varepsilon/2$, so $\Delta(f,P) < \varepsilon$. ■

As a corollary, we obtain:

Theorem 6.1.5. *Let f be a bounded function on $[a.b]$, and let (P_n) be a sequence of partitions of $[a,b]$ with* $\operatorname{mesh} P_n \longrightarrow 0$. *Then f is integrable on $[a,b]$ iff $\Delta(f,P_n) \longrightarrow 0$, in which case $U(f,P_n), L(f,P_n) \longrightarrow \int_a^b f$, and also for sequence (σ_n) of selections σ_n subordinate to P_n, $S(f,P_n,\sigma_n) \longrightarrow \int_a^b f$.*

Example. Let $f(x) = x$, $x \in [0,1]$. For every $n \in \mathbb{N}$ let $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1\right\}$. Then for every n, for every $1 \leq i \leq n$, $\Delta x_i = \frac{1}{n}$, $M_i = \frac{i}{n}$, $m_i = \frac{i-1}{n}$, $\operatorname{Var}_{[x_{i-1},x_i]} f = M_i - m_i = \frac{1}{n}$, so

$$U(f,P_n) = \sum_{i=1}^n \tfrac{i}{n} \cdot \tfrac{1}{n} = \tfrac{1}{n^2}\sum_{i=1}^n i = \tfrac{1}{n^2} \cdot \tfrac{n(n+1)}{2} = \left(1 + \tfrac{1}{n}\right)\tfrac{1}{2},$$

$$L(f,P_n) = \sum_{i=1}^n \tfrac{i-1}{n} \cdot \tfrac{1}{n} = \tfrac{1}{n^2}\sum_{i=1}^n (i-1) = \tfrac{1}{n^2} \cdot \tfrac{n(n-1)}{2} = \left(1 - \tfrac{1}{n}\right)\tfrac{1}{2},$$

$$\Delta(f,P_n) = \sum_{i=1}^n \tfrac{1}{n} \cdot \tfrac{1}{n} = \tfrac{1}{n}.$$

As $\Delta(f,P_n) \longrightarrow 0$ (or since $U(f,P_n)$ and $L(f,P_n)$ converge to the same limit), f is integrable, and $\int_0^1 f = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n) = \frac{1}{2}$.

Actually, all continuous and all monotone functions are integrable:

Theorem 6.1.6. *If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.*

Proof. If f is continuous on $[a,b]$ then f is bounded and uniformly continuous on $[a,b]$. Let $\varepsilon > 0$, find $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/(b-a)$ whenever $x, y \in [a,b]$, $|x-y| < \delta$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be an partition of $[a,b]$ with $\operatorname{mesh} P < \delta$, then for any i, $\operatorname{Var}_{[x_{i-1},x_i]} f \le \varepsilon/(b-a)$, so

$$\Delta(f,P) = \sum_{i=1}^{n} \operatorname{Var}_{[x_{i-1},x_i]} f \Delta x_i \le \sum_{i=1}^{n} \tfrac{\varepsilon}{b-a} \Delta x_i = \tfrac{\varepsilon}{b-a} \sum_{i=1}^{n} \Delta x_i = \tfrac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

■

Theorem 6.1.7. *If f is monotone on $[a,b]$, then f is integrable on $[a,b]$.*

Proof. W.l.o.g. assume that f is increasing; then f is bounded below by $f(a)$ and above by $f(b)$. Also assume that $f(a) \ne f(b)$, otherwise f is constant. Given $\varepsilon > 0$, put $\delta = \varepsilon/(f(b) - f(a))$. Now if $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$ with $\operatorname{mesh} P < \delta$, then

$$\Delta(f,P) = \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big)\Delta x_i \le \Big(\sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big)\Big)\delta = (f(b) - f(a))\delta = \varepsilon.$$

■

If a function is discontinuous at only finitely many points, it is also integrable. Moreover, let's denote by $\operatorname{Disc}(f)$ the set of discontinuity of f, that is, the set of points at which f is discontinuous; it is easy to see that the following is true:

Theorem 6.1.8. *Let f be a bounded function on an interval $[a,b]$ with the property that for any $\delta > 0$, $\operatorname{Disc}(f)$ is contained in a finite union of open intervals of total length $< \delta$. Then f is integrable on $[a,b]$.*

Proof. Let $\varepsilon > 0$. Let $|f| \le M$; put $\delta = \varepsilon/(4M)$. Let $I_1, \ldots, I_n$ be open subintervals of $[a,b]$ such that $\operatorname{Disc}(f) \subseteq \bigcup_{i=1}^{n} I_i$ and $\sum_{i=1}^{n} |I_i| < \delta$; we may assume that these intervals are disjoint. The complement $[a,b] \setminus \bigcup_{i=1}^{n} I_i$ is a union of closed intervals; let $J_1, \ldots, J_k$ be these intervals. For each j, f is continuous on J_j, thus there is a partition P_j of J_j such that $\Delta(f, P_j) < \varepsilon/(2k)$. Let $P = \bigcup_{j=1}^{k} P_j$. (The intervals defined by P are the subintervals of J_j defined by P_j, $j = 1, \ldots, k$, and the (closures of) intervals $I_1, \ldots, I_n$.) Since the intervals $I_1, \ldots, I_n$ contribute to $\Delta(f,P)$ at most $2M\delta$, we have $\Delta(f,P) = \sum_{j=1}^{k} \Delta(f,P_j) + 2M\delta < k\varepsilon/(2k) + \varepsilon/2 = \varepsilon$. ■

Example. An example of a set that can be covered by finitely many intervals of arbitrarily small total length is Cantor's set C. Hence, *the indicator function* $f(x) = \begin{cases} 1, & x \in C \\ 0, & x \notin C \end{cases}$, for which $\operatorname{Disc}(f) = C$, is integrable on $[0,1]$ (with $\int f = 0$).

Theorem 6.1.8 is not a *criterion* of integrability, as it gives a sufficient but not a necessary condition: Riemann's function $\begin{cases} 0, & x \notin \mathbb{Q} \\ \frac{1}{m}, & x = \frac{n}{m} \in \mathbb{Q} \end{cases}$ is integrable on $[0,1]$ though its set of discontinuity, which is $\mathbb{Q} \cap [0,1]$, cannot be covered by finitely many intervals of length $1/2$. It can however be covered by a countable set of intervals of arbitrarily small total length: let $\varepsilon > 0$, enumerate $\mathbb{Q} \cap [0,1] = \{a_1, a_2, \ldots\}$, for each i take I_i be the interval $\big(a_i - \frac{\varepsilon}{2^{i+1}}, a_i + \frac{\varepsilon}{2^{i+1}}\big)$, then $\sum_{i=1}^{\infty} |I_i| = \varepsilon$. (This infinite sum is called *a series*, we will study series soon.)

The actual criterion of integrability is *Lebesgue's criterion*. A set $A \subseteq \mathbb{R}$ is said to be *a null-set*, or to have *zero measure*, if it can be covered by countably many (open) intervals of arbitrarily small total length: for any $\varepsilon > 0$ there is a sequence $I_1, I_2, \ldots$ of open intervals such that $A \subseteq \bigcup_{i=1}^{\infty} I_i$ and $\sum_{i=1}^{\infty} |I_i| < \varepsilon$. Every finite or countable set is a null-set; the (uncountable!) Cantor set is also a null-set.

Theorem 6.1.9. (Lebesgue's criterion) *A bounded function f on a bounded interval is integrable iff* $\operatorname{Disc}(f)$ *is a null-set.*

We will not prove this criterion (though the proof is not very difficult).

Let's obtain another, more elementary criterion:

Theorem 6.1.10. *A bounded function f is integrable on $[a,b]$ iff for any $\tau, \delta > 0$ there is a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ such that $\sum_{i:\operatorname{Var}_{[x_{i-1},x_i]} f \ge \tau} \Delta x_i < \delta$.*

Proof. Assume that for any $\tau, \delta > 0$ there is a partition $P = \{x_1, x_1, \ldots, x_n\}$ of $[a,b]$ such that $\sum_{i:\mathrm{Var}_{[x_{i-1},x_i]} f \geq \tau} \Delta x_i < \delta$. Let $\varepsilon > 0$, let $|f| \leq M$. Find a pertition $P = \{x_1, x_1, \ldots, x_n\}$ of $[a,b]$ such that for $K = \bigl\{i : \mathrm{Var}_{[x_{i-1},x_i]} f \geq \frac{\varepsilon}{2(b-a)}\bigr\}$ we have $\sum_{i\in K} \Delta x_i < \frac{\varepsilon}{4M}$. Then

$$\Delta(f,P) = \sum_{i\in K} \mathrm{Var}_{[x_{i-1},x_i]} f \Delta x_i + \sum_{i\notin K} \mathrm{Var}_{[x_{i-1},x_i]} f \Delta x_i < 2M \sum_{i\in K} \Delta x_i + \tfrac{\varepsilon}{2(b-a)} \sum_{i\notin K} \Delta x_i$$
$$< 2M\frac{\varepsilon}{4M} + \frac{\varepsilon}{2(b-a)}(b-a) = \varepsilon.$$

Hence, f is integrable.

Now assume that there are $\tau, \delta > 0$ such that for any partition $P = \{x_1, x_1, \ldots, x_n\}$ of $[a,b]$, $\sum_{i:\mathrm{Var}_{[x_{i-1},x_i]} f \geq \tau} \Delta x_i \geq \delta$. Then for any partition $P = \{x_1, x_1, \ldots, x_n\}$, taking $K = \bigl\{i : \Delta_{[x_{i-1},x_i]} f \geq \tau\bigr\}$ we have $\Delta(f,P) \geq \sum_{i\in K} \mathrm{Var}_{[x_{i-1},x_i]} f \Delta x_i \geq \tau \sum_{i\in K} \Delta x_i \geq \tau\delta$. Hence, f is not integrable. ■

6.2. Properties of the integral

Theorem 6.2.1. *Let $a < b < c$ and let f be a bounded function on $[a,c]$. Then f is integrable on $[a,c]$ iff f is integrable on both $[a,b]$ and $[b,c]$, in which case $\int_a^c f = \int_a^b f + \int_b^c f$.*

Proof. For a partition P of $[a,c]$ with $b \in P$ let $P' = P \cap [a,b]$ and $P'' = P \cap [b,c]$, then clearly $U(f,P) = U(f,P') + U(f,P'')$, $L(f,P) = L(f,P') + L(f,P'')$, and $\Delta(f,P) = \Delta(f,P') + \Delta(f,P'')$.

If f is integrable on $[a,c]$ then for any $\varepsilon > 0$ there is a partition P for which $\Delta(f,P) < \varepsilon$, then $\Delta(f,P'), \Delta(f,P'') < \varepsilon$, so f is integrable on $[a,b]$ and on $[b,c]$. If f is integrable on $[a,b]$ and on $[b,c]$ then for any $\varepsilon > 0$ there are partitions P' of $[a,b]$ and P'' of $[b,c]$ such that $\Delta(f,P'), \Delta(f,P'') < \varepsilon/2$, then for $P = P' \cup P''$, $\Delta(f,P) < \varepsilon$, so f is integrble on $[a,c]$.

And in this case, since $0 \leq U(f,P) - \int_a^c f < \varepsilon$, $0 \leq U(f,P') - \int_a^b f < \varepsilon/2$ and $0 \leq U(f,P'') - \int_b^c f < \varepsilon/2$, we have $\bigl|\int_a^c f - \bigl(\int_a^b f + \int_b^c f\bigr)\bigr| < \varepsilon$. Since this is true for any $\varepsilon > 0$, $\int_a^c f = \int_a^b f + \int_b^c f$. ■

Theorem 6.2.2. (i) *If f is integrable on $[a,b]$, then for any $c \in \mathbb{R}$, cf is integrable on $[a,b]$ and $\int_a^b (cf) = c\int_a^b f$.*

(ii) *If f, g are integrable on $[a,b]$, then $f+g$ is integrable on $[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.*

(iii) *If f, g are integrable on $[a,b]$, then fg is integrable on $[a,b]$.*

(iv) *If f is integrable on $[a,b]$ and φ is a bounded continuous function on* $\mathrm{Rng}(f)$*, then $\varphi\circ f$ is integrable on $[a,b]$.*

Proof. (i) If $c = 0$ the statement is trivial. If $c > 0$ then for any interval I, $\sup\bigl\{cf(x),\ x \in I\bigr\} = c\sup\bigl\{f(x),\ x \in I\bigr\}$ and $\inf\bigl\{cf(x),\ x \in I\bigr\} = c\inf\bigl\{f(x),\ x \in I\bigr\}$; if $c < 0$ then for any interval I, $\sup\bigl\{cf(x),\ x \in I\bigr\} = c\inf\bigl\{f(x),\ x \in I\bigr\}$ and $\inf\bigl\{cf(x),\ x \in I\bigr\} = c\sup\bigl\{f(x),\ x \in I\bigr\}$. Thus if $c > 0$, then for any partition P of $[a,b]$, $U(cf,P) = cU(f,P)$ and $L(cf,P) = cL(f,P)$; if $c < 0$, then $U(cf,P) = cL(f,P)$ and $L(cf,P) = cU(f,P)$; in both cases, $\Delta(cf,P) = |c|\Delta(f,P)$. So, given $\varepsilon > 0$, if P is such that $\Delta(f,P) < \varepsilon/|c|$, then $\Delta(cf,P) < \varepsilon$. So, cf is integrable. Also, for such P, $\bigl|U(cf,P) - \int_a^b cf\bigr| < \varepsilon$ and $\bigl|cU(f,P) - c\int_a^b f\bigr| = |c|\bigl|U(f,P) - \int_a^b f\bigr| < |c|\varepsilon/|c| = \varepsilon$, so $\bigl|\int_a^b cf - c\int_a^b f\bigr| < 2\varepsilon$. Since this is true for all $\varepsilon > 0$, $\int_a^b (cf) = c\int_a^b f$.

(ii) For any interval I, $\sup\bigl\{f(x)+g(x),\ x \in I\bigr\} \leq \sup\bigl\{f(x),\ x \in I\bigr\} + \sup\bigl\{g(x),\ x \in I\bigr\}$ and $\inf\bigl\{f(x) + g(x),\ x \in I\bigr\} \leq \inf\bigl\{f(x),\ x \in I\bigr\} + \inf\bigl\{g(x),\ x \in I\bigr\}$, so for any partition P of $[a,b]$, $U(f+g,P) \leq U(f,P) + U(g,P)$ and $L(f+g,P) \geq L(f,P) + L(g,P)$, and so, $\Delta(f+g,P) \leq \Delta(f,P) + \Delta(g,P)$. Given $\varepsilon > 0$, if P is such that $\Delta(f,P), \Delta(g,P) < \varepsilon/2$, then $\Delta(f+g,P) < \varepsilon$. So, $f+g$ is integrable. Also, using this partition we see that $\bigl|\int_a^b (f+g) - \bigl(\int_a^b f + \int_a^b g\bigr)\bigr| < 2\varepsilon$; since this is true for all $\varepsilon > 0$, $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

(iii) follows from Lebesgue's criterion of integrability, since $\operatorname{Disc}(fg) \subseteq \operatorname{Disc}(f) \cup \operatorname{Disc}(g)$. Without that criterion, we can use the following inequality: if M is such that $|f|, |g| \le M$, then for any subinterval I of $[a, b]$, $\operatorname{Var}_I(fg) \le M(\operatorname{Var}(f) + \operatorname{Var}(g))$. Indeed, for any $x, y \in I$,

$$|f(x)g(x) - f(y)g(y)| \le |f(x)g(x) - f(y)g(x)| + |f(y)g(x) - f(y)g(y)| \le M\big(|f(x) - f(y)| + |g(x) - g(y)|\big),$$

so

$$\begin{aligned}\operatorname{Var}_I(fg) &= \sup\big\{|f(x)g(x) - f(y)g(y)|,\ x, y, \in I\big\} \\ &\le M\Big(\sup\big\{|f(x) - f(y)|,\ x, y, \in I\big\} + \sup\big\{|g(x) - g(y)|,\ x, y, \in I\big\}\Big) = M\big(\operatorname{Var}_I(f) + \operatorname{Var}_I(g)\big).\end{aligned}$$

It follows that for any partition P of $[a, b]$, $\Delta(fg, P) \le M\big(\Delta(f, P) + \Delta(g, P)\big)$, and if f and g are integrable, then so is fg.
(iv) also easily follows from Lebesgue's criterion: if φ is continuous, then $\operatorname{Disc}(\varphi \circ f) \subseteq \operatorname{Disc}(f)$. I'll use Theorem 6.1.10 instead, but ASSUME, for simplicity, that φ is UNIFORMLY continuous. Let $\tau, \delta > 0$. Find $\varepsilon > 0$ such that for $u, v \in \operatorname{Rng}(f)$, $|u - v| < \varepsilon$ implies that $|\varphi(u) - \varphi(v)| < \tau$. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$ such that $\sum_{i:\operatorname{Var}_{[x_{i-1},x_i]} f \ge \varepsilon} \Delta x_i < \delta$. For every i for which $\operatorname{Var}_{[x_{i-1},x_i]} f < \varepsilon$, for any $y, z \in [x_{i-1}, x_i]$ we have $|f(y) - f(z)| < \varepsilon$, so $\big|\varphi(f(y)) - \varphi(f(z))\big| < \tau$; hence, $\big\{i : \operatorname{Var}_{[x_{i-1},x_i]}(\varphi \circ f) \ge \tau\big\} \subseteq \big\{i : \operatorname{Var}_{[x_{i-1},x_i]} f \ge \varepsilon\big\}$. So, $\sum_{i:\operatorname{Var}_{[x_{i-1},x_i]}(\varphi \circ f) \ge \tau} \Delta x_i < \delta$, and $\varphi \circ f$ is integrable by Theorem 6.1.10. ■

Theorem 6.2.3. *If f and g are integrable on $[a, b]$ and $f \le g$ on $[a, b]$, then $\int_a^b f \le \int_a^b g$.*

Proof. Clearly, for any partition P of $[a, b]$, $U(f, P) \le U(g, P)$, so $\int_a^b f = U(f) \le U(g) = \int_a^b g$. ■

In particular, we have

Corollary 6.2.4. *Let f be integrable on $[a, b]$; if $f \le M$, then $\int_a^b f \le M(b - a)$; if $f \ge m$, then $\int_a^b f \ge m(b - a)$.*

6.3. Mean value theorems for integrals

Like for derivatives, there are two mean value theorems for integrals:

Theorem 6.3.1. *If f is continuous on $[a, b]$ then there is $c \in [a, b]$ such that $\int_a^b f = f(c)(b - a)$.*

Proof. Let M and m be the maximal and the minimal values of f on $[a, b]$ respectively. Then $m(b - a) \le \int_a^b f \le M(b - a)$, so $m \le \int_a^b f/(b - a) \le M$. Since f is continuous, by the I.V.T. there exists $c \in [a, b]$ such that $f(c) = \int_a^b f/(b - a)$. ■

The second mean value theorem for integrals is

Theorem 6.3.2. *If f is continuous on $[a, b]$, g is integrable on $[a, b]$ and $g \ge 0$ or $g \le 0$, then there is $c \in [a, b]$ such that $\int_a^b (fg) = f(c) \int_a^b g$.*

(The condition $g \ge 0$ or $g \le 0$ is essential: for $f(x) = g(x) = x$ on $[-1, 1]$ we have $\int_{-1}^1 g = 0$ and $\int_{-1}^1 fg = 2/3$, so there is no c such that $\int_a^b (fg) = f(c) \int_a^b g$.)

Proof. W.l.o.g. assume that $g \ge 0$, then $\int_a^b g \ge 0$ as well. Let M and m be the maximal and the minimal values of f on $[a, b]$ respectively, then $mg \le fg \le Mg$, so

$$m \int_a^b g = \int_a^b mg \le \int_a^b fg \le \int_a^b Mg = M \int_a^b g.$$

The continuous function $h(x) = f(x) \int_a^b g$ takes the values $m \int_a^b g$ and $M \int_a^b g$, so by the I.V.T. there exists $c \in [a, b]$ such that $h(c) = \int_a^b fg$. ■

6.4. Some integral inequalities

Integration is "an infinite summation (of infinitely small numbers)", and many inequalities known for sums can be generalized to integrals.

The triangle inequality $|u+v| \le |u|+|v|$ takes the following form:

Theorem 6.4.1. *For any function f integrable on $[a,b]$, $\left|\int_a^b f\right| \le \int_a^b |f|$.*

Proof. Since $-|f| \le f \le |f|$, $-\int_a^b |f| = \int_a^b -|f| \le \int_a^b f \le \int_a^b |f|$, so $\left|\int_a^b f\right| \le \int_a^b |f|$. ■

Under a *triangle inequality* we can also understand the following easy fact:

Theorem 6.4.2. *For any functions f and g integrable on $[a,b]$, $\int_a^b |f+g| \le \int_a^b |f| + \int_a^b |g|$.*

The Cauchy-Schwarz inequality for integrals is:

Theorem 6.4.3. *For any two functions f and g integrable on $[a,b]$, $\left(\int_a^b fg\right)^2 \le \int_a^b f^2 \cdot \int_a^b g^2$.*

Proof. The proof is the same as for ordinary sums: Consider the quadratic polynomial $h(t) = \int_a^b (tf-g)^2 = t^2\int_a^b f^2 - 2t\int_a^b fg + \int_a^b g^2$. We have $h(t) \ge 0$ for all t, so the discrimiant $\left(-2\int_a^b fg\right)^2 - 4\int_a^b f^2 \int_a^b g^2 \le 0$. ■

Here is another 'triangle inequality", that corresponds to Theorem 1.6.3:

Theorem 6.4.4. *For any functions f and g integrable on $[a,b]$, $\left(\int_a^b (f+g)^2\right)^{1/2} \le \left(\int_a^b f^2\right)^{1/2} + \left(\int_a^b g^2\right)^{1/2}$.*

Proof. The square of the left-hand part of the inequality is $\int_a^b (f+g)^2 = \int_a^b f^2 + \int_a^b g^2 + 2\int_a^b fg$ and of the right-hand part is $\int_a^b f^2 + \int_a^b g^2 + 2\left(\int_a^b f^2\right)^{1/2}\left(\int_a^b g^2\right)^{1/2}$. Since $\left|\int_a^b fg\right| \le \left(\int_a^b f^2\right)^{1/2}\left(\int_a^b g^2\right)^{1/2}$ by the Cauchy-Schwarz inequality, the left-hand part is $\le$ right-hand part. ■

Here is a version of Jensen's inequality for integrals:

Theorem 6.4.5. *Let φ be a convex function on an open interval I and let $f\colon [a.b] \longrightarrow I$ be an integrable function. Then $\varphi\left(\frac{1}{b-a}\int_a^b f\right) \le \frac{1}{b-a}\int_a^b \varphi\circ f$.*

Proof. Let $P = \{x_0,\dots,x_n\}$ be a partition of $[a,b]$ and let $\sigma = \{z_1,\dots,z_n\}$ be a selection subordinate to P. Then $\frac{1}{b-a}S(f,P,\sigma) = \sum_{i=1}^n f(z_i)\frac{\Delta x_i}{b-a}$ and $\frac{1}{b-a}S(\varphi\circ f,P,\sigma) = \sum_{i=1}^n \varphi(f(z_i))\frac{\Delta x_i}{b-a}$. The numbers $t_i = \frac{\Delta x_i}{b-a}$ are >0 and satisfy $\sum_{i=1}^n t_i = 1$; hence, by Jensen's inequality or numbers (Theorem 5.2.3) we have

$$\varphi\left(\tfrac{1}{b-a}S(f,P,\sigma)\right) \le \tfrac{1}{b-a}S(\varphi\circ f,P,\sigma).$$

Now let (P_n) be a sequence of partitions of $[a,b]$ such that $\Delta(f,P_n), \Delta(\varphi\circ f,P_n) \longrightarrow 0$ as $n \longrightarrow \infty$, and for every n let σ_n be a selection subordinate to P_n; then $\frac{1}{b-a}S(f,P_n,\sigma_n) \longrightarrow \frac{1}{b-a}\int_a^b f$ and $\frac{1}{b-a}S(\varphi\circ f,P_n,\sigma_n) \longrightarrow \frac{1}{b-a}\int_a^b \varphi\circ f$. Since φ is continuous, we therefore also have

$$\lim_{n\to\infty} \varphi\left(\tfrac{1}{b-a}S(f,P_n,\sigma_n)\right) = \varphi\Big(\lim_{n\to\infty} \tfrac{1}{b-a}S(f,P_n,\sigma_n)\Big) = \varphi\left(\tfrac{1}{b-a}\int_a^b f\right).$$

Since for every n, $\varphi\left(\frac{1}{b-a}S(f,P_n,\sigma_n)\right) \le \frac{1}{b-a}S(\varphi\circ f,P_n,\sigma_n)$, we obtain that $\varphi\left(\frac{1}{b-a}\int_a^b f\right) \le \frac{1}{b-a}\int_a^b \varphi\circ f$. ■

6.5. The fundamental theorem of calculus

Let I be an interval, bounded or unbounded. A function $f\colon I \longrightarrow \mathbb{R}$ (not necessarily bounded) is said to be *locally integrable* on I if f is integrable over every closed bounded subinterval of I.

For $a,b \in I$ with $b<a$ we define $\int_b^a f = -\int_a^b f$, and for $b=a$, $\int_a^b f = 0$. Then for any $a,b,c \in I$, $\int_a^c f = \int_a^b f + \int_b^c f$: if, say, $c<a<b$, then $\int_c^b f = \int_c^a f + \int_a^b f$, so $\int_a^c f = -\int_c^a f = \int_a^b f - \int_c^b f = \int_a^b f + \int_b^c f$.

Let f be locally integrable on I. Fix $a \in I$; the function $F(x) = \int_a^x f = \int_a^x f(t)\,dt$, $x \in I$, is called *an integral function* of f. For any $x,y \in I$ we then have $F(y)-F(x) = \int_a^y f - \int_a^x f = \int_x^y f$. (Notice that any other inegral function $G(x) = \int_b^x f$, for $b \in I$, satisfies $G(x) = F(x) + \int_b^a f = F(x) + \text{const.}$)

Theorem 6.5.1. *Let f be locally integrable on I and F be an integral function of f. Then F is continuous on I. If f is bounded on I, $|f| \leq M$, then F is Lipschitz on I with constant M, $|F(x) - F(y)| \leq M|x - y|$ for all $x, y \in I$.*

Proof. If $|f| \leq M$ on I, then for any $x, y \in I$ with $x < y$,

$$|F(y) - F(x)| = \Big|\int_x^y f\Big| \leq \int_x^y |f| \leq \int_x^y M = M(y - x),$$

so F is Lipschitz on I.

In the general case, where f is not bounded on I, f is integrable and so bounded on any closed bounded subinterval J of I, so F is Lipschitz and so continuous on J. Hence, F is continuous at every point of I. ■

Theorem 6.5.2. *Let f be locally integrable on an interval I, let F be an integral function of f, let $x_0 \in I$. If f is continuous at x_0 then F is differentiable at x_0 and $F'(x_0) = f(x_0)$. If f is left-continuous (respectively, right-continuous) at x_0, then F is left-hand (respectively, right-hand) differentiable at x_0 with $F'_-(x_0) = f(x_0)$ or, respectively, $F'_+(x_0) = f(x_0)$.*

Proof. I'll only prove this for the right-hand derivative. For any $x \in I$ we have $\frac{F(x)-F(x_0)}{x-x_0} = \frac{\int_{x_0}^x f}{x-x_0}$. Let f be right-continuous at x_0. Let $\varepsilon > 0$, find $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in [x_0, x_0 + \delta)$. Let $x_0 < x < x_0 + \delta$, then for any $t \in [x_0, x]$, $f(x_0) - \varepsilon < f(t) < f(x_0) + \varepsilon$, so

$$(f(x_0) - \varepsilon)(x - x_0) \leq \int_{x_0}^x f \leq (f(x_0) + \varepsilon)(x - x_0),$$

so $f(x_0) - \varepsilon \leq \frac{\int_{x_0}^x f}{x-x_0} \leq f(x_0) + \varepsilon$. So, $F'_+(x_0) = \lim_{x\to 0^+} \frac{F(x)-F(x_0)}{x-x_0} = \lim_{x\to 0^+} \frac{\int_{x_0}^x f}{x-x_0} = f(x_0)$. ■

F does not change if the value of f is changed at just one point, so the assumption that f is (left, right) continuous at x_0 can be replaced by the assumption that f has a (left-hand, right-hand) limit at x_0.

A function G is said to be *a primitive*, or *an antiderivative* of f, if $G' = f$. As we know, if G is a primitive of f on an interval I, then another function H is a primitive of f on I iff $H = G + \text{const}$.

From Theorem 6.5.2 we obtain (the simplest version of) *the Fundamental Theorem of Calculus, F.T.C.*:

Theorem 6.5.3. *If f is a continuous function on an interval I, then its integral function F is its primitive, $F' = f$.*

There is a more sophisticated version of the F.T.C.:

Theorem 6.5.4. *If a function G is differentiable on an interval I, $f = G'$ is locally integrable on I, and F is an integral function of f, then $F = G + \text{const}$.*

Proof. Let $a \in I$ and $F(x) = \int_a^x f(t)\,dt$, $x \in I$. Fix $x > a$. Let $P = \{a = x_0 < x_1 < \ldots, x_n = x\}$ be a partition of $[a, x]$. By the M.V.T., for every i, $G(x_i) - G(x_{i-1}) = f(z_i)\Delta x_i$ for some $z_i \in (x_{i-1}, x_i)$. Let $\sigma_P = \{z_1, \ldots, z_n\}$; then $S(f, P, \sigma_P) = \sum_{i=1}^n f(z_i)\Delta x_i = \sum_{i=1}^n (G(x_i) - G(x_{i-1})) = G(x) - G(a)$. When $\operatorname{mesh} P \longrightarrow 0$, $S(f, P, \sigma_P) \longrightarrow \int_a^x f = F(x)$. So, $F(x) = G(x) - G(a)$.

For $x < a$, the proof is similar. (Alternatively, we may apply the already proved result to $G(-x)$.) ■

Here is another, cumulative formulation of F.T.C.:

Theorem 6.5.5. *If function f has a primitive on an interval I and is locally integrable on I, then an integral function F of f is also a primitive of f. If f is continuous on I, then it does have a primitive (and is locally integrable).*

Corollary 6.5.6. *If f is integrable on $[a, b]$ and has a primitive G on $[a, b]$, then $\int_a^b f = G(b) - G(a)$.*

Proof. Let F be an integral function of f on $[a, b]$, then $F = G+\text{const}$, so $\int_a^b f = F(b)-F(a) = G(b)-G(a)$. ■

6.6. Techniques of integration

Given a locally integrable function f on an interval I, *the indefinite integral* $\int f$, or $\int f(x)\,dx$, is a (any) primitive function of f, if it exists. If F is such a primitive, then for any $a,b \in I$ we have $\int_a^b f = F(b) - F(a)$, which is also denoted by $F\big|_a^b$.

If f is an elementary function, its primitive may not be elementary; in this case we say that f *is non-integrable in elementary terms*. (Such are functions e^{x^2}, e^x/x, and $\sin x/x$, for example.) If a function f is integrable in elementary terms, the process of finding an expression for the primitive of f in terms of power/exponential/trigonometric functions is called *integration*.

While differentiation is algorithmic and straightforward, integration is an art. You are supposed to know primitives of the most common functions, and then apply some tools to reduce the integral under consideration to these "table" integrals.

Actually, "the art of integration" uses only three simple tools. The first one is trivial:

Theorem 6.6.1. (i) *Let function f have primitives on an interval I, $f = F'$. Then for any $c \in \mathbb{R}$, the function cf also has a primitive on I, which is $\int (cf) = cF$.*
(ii) *Let functions f and g have primitives on an interval I, $f = F'$ and $g = G'$. Then the function $f+g$ also has a primitive on I, which is $\int (f+g) = F + G$.*

The second tool is based on the identity $(FG)' = F'G + FG'$; it is called *integration by parts*:

Theorem 6.6.2. *Let functions f and g have primitives on an interval I, $f = F'$ and $g = G'$, and suppose that the function Fg also has a primitive on I. Then fG has a primitive, which is $\int fG = FG - \int Fg$. If, additionally, f and g are integrable on an interval $[a,b]$, then $\int_a^b fG = FG\big|_a^b - \int_a^b Fg$.*

There is a convenient notation, which I highly recommend: $dF(x) = F'(x)\,dx$. In this notation, the integration by parts formula takes the form $\int G(x)\,dF(x) = F(x)G(x) - \int F(x)\,dG(x)$ (or just $\int G\,dF = FG - \int F\,dG$).

The third tool is based on the chain rule $(F(\varphi(t))' = F'(\varphi(t))\varphi'(t)$; it is called *substitution*:

Theorem 6.6.3. *Let function φ be differentiable on an interval I and function f have a primitive on $\varphi(I)$, $f = F'$. Then the function $(f\circ\varphi)\varphi'$ also has a primitive on I, which is $\int (f\circ\varphi)\varphi' = F\circ\varphi$. If, additionally, $(f\circ\varphi)\varphi'$ is integrable on an interval $[a,b]$ and f is integrable on $\varphi([a,b])$, then $\int_a^b (f\circ\varphi)\varphi' = \int_{\varphi(a)}^{\varphi(b)} f$.*

In the notation $dF(x) = F'(x)dx$ the substitution formula takes the form $\int f(\varphi(t))\,d\varphi(t) = F(\varphi(t))$, that is, $\int f(\varphi(t))\,d\varphi(t) = \int f(x)\,dx\big|_{x=\varphi(t)}$.

Examples. (i) Integraton by parts: $\int x\sin x\,dx = -\int x\,d\cos x = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C$, where C is any constant. (And indeed, $(-x\cos x + \sin x)' = -\cos x + x\sin x + \cos x = x\sin x$.)
(ii) Substitution $y = x^2$: $\int xe^{x^2}\,dx = \frac{1}{2}\int e^{x^2}dx^2 = \frac{1}{2}\int e^y\,dy = \frac{1}{2}e^y + C = \frac{1}{2}e^{x^2} + C$.
(iii) Substitution $x = \sin t$: $\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2 t}\,d\sin t = \int \cos t\cos t\,dt = \frac{1}{2}\int (1+\cos 2t)\,dt = \frac{1}{2}t + \frac{1}{4}\sin 2t + C = \frac{1}{2}(t + \sin t\cos t) + C = \frac{1}{2}\big(\arcsin x + x\sqrt{1-x^2}\big) + C$.
(iv) The following example shows how our usual carelessness in substitutions can lead to mistakes. Using the substitution $y = \sin x$ we compute

$$\begin{aligned}\int \sqrt{1+\sin x}dx = \int \sqrt{1+y}d\arcsin y = \int \frac{\sqrt{1+y}}{\sqrt{1-y^2}}dy = \int \frac{dy}{\sqrt{1-y}} = \int \frac{-d(1-y)}{\sqrt{1-y}} &= -2\sqrt{1-y} \\ &= -2\sqrt{1-\sin x} + C.\end{aligned}$$

But this is strange: the function $\sqrt{1+\sin x}$ is defined and nonnegative on $\mathbb{R}$, so its primitive has to be increasing, but the function $-2\sqrt{1-\sin x}$ is periodic and cannot be increasing on $\mathbb{R}$. The mistake was that $y = \sin x$ implies $x = \arcsin y$ only for $x \in \left[\frac{\pi}{2}, \frac{-\pi}{2}\right]$. And indeed, we have

$$\big(-2\sqrt{1-\sin x}\big)' = \frac{\cos x}{\sqrt{1-\sin x}} = \frac{\sqrt{1-\sin^2 x}}{\sqrt{1-\sin x}} = \sqrt{1+\cos x}$$

iff $\cos x = \sqrt{1-\sin^2 x}$, that is, iff $\cos x \geq 0$, that is, iff $x \in \left[2k\pi - \frac{\pi}{2}, 2k\pi + \frac{\pi}{2}\right]$ for some $k \in \mathbb{Z}$. On the intervals $\left[2k\pi + \frac{\pi}{2}, 2k\pi + \frac{3\pi}{2}\right]$, $k \in \mathbb{Z}$, where $\cos x = -\sqrt{1-\sin^2 x}$, the primitives of $\sqrt{1+\sin x}$ are the functions $2\sqrt{1-\sin x} + C$.

To construct a "global" primitive F of $\sqrt{1+\sin x}$ on $\mathbb{R}$ we define it to be $-2\sqrt{1-\sin x}+a_k$ for $x \in \left[2k\pi - \frac{\pi}{2}, 2k\pi + \frac{\pi}{2}\right]$, $k \in \mathbb{Z}$, and $2\sqrt{1-\sin x}+b_k$ for $x \in \left[2k\pi + \frac{\pi}{2}, 2k\pi + \frac{3\pi}{2}\right]$, $k \in \mathbb{Z}$, where a_k and b_k must be chosen so that F is continuous. At every point $x_k = 2k\pi + \pi/2$, $k \in \mathbb{Z}$, we have $F(x_k) = -2\sqrt{1-\sin x_k} + a_k = a_k$ and also $F(x_k) = 2\sqrt{1-\sin x_k} + b_k = b_k$, so $b_k = a_k$. At every point $y_k = 2k\pi + 3\pi/2 = 2(k+1)\pi - \pi/2$, $k \in \mathbb{Z}$, we have $F(y_k) = 2\sqrt{1-\sin y_k} + b_k = b_k + 2\sqrt{2}$ and also $F(y_k) = -2\sqrt{1-\sin y_k} + a_{k+1} = a_{k+1} - 2\sqrt{2}$, so $a_{k+1} = b_k + 4\sqrt{2}$. Hence if we put $a_0 = 0$, we must have $a_k = b_k = 4k\sqrt{2}$ for all $k \in \mathbb{Z}$.

6.7. Improper integrals

The area under the graph of a function on an interval can be finite even if the function or the interval are unbounded. The definition of Riemann integral, via partitions, fails in these situation, but we can find this area as the limit of "conventional" itegrals.

Let $I = (\alpha, \beta)$ be an interval, where α and β can be numbers, α can be $-\infty$, and β can be $+\infty$. Let f be a locally integrable (possibly unbounded) function on (α, β). *The improper integral* $\int_\alpha^\beta f$ is defined as $\lim_{\substack{a\to\alpha\\ b\to\beta}} \int_a^b f$; if F is an integral function of f, $\int_\alpha^\beta f = \lim_{\substack{a\to\alpha\\ b\to\beta}} (F(b) - F(a))$. If this limit exists and is finite, we say that the integral $\int_\alpha^\beta f$ *converges*; if the limit is infinite, we say that the integral *diverges to* ∞; if the limit doesn't exist, we say that the integral *diverges*.

Examples. (i) $\int_0^{+\infty} \sin x\, dx$ diverges: $\int_0^b \sin x\, dx = -\cos x\big|_0^b = -\cos b + 1$ has no limit as $b \longrightarrow +\infty$.

(ii) $\int_0^{+\infty} e^{-x} dx = \lim_{b\to+\infty} \int_0^b e^{-x} dx = \lim_{b\to+\infty} -e^{-x}\big|_1^b = \lim_{b\to+\infty} (-e^{-b}) + e^0 = 1$.

(iii) $\int_1^{+\infty} \frac{dx}{x}$ diverges to $+\infty$: $\int_1^b \frac{dx}{x} = \log x\big|_1^b = \log b \longrightarrow +\infty$ as $b \longrightarrow +\infty$.

(iv) For any $p > 1$, $\int_1^{+\infty} \frac{dx}{x^p}$ convegres to $\frac{1}{p-1}$: $\int_1^b \frac{dx}{x^p} = \frac{-1}{(p-1)x^{p-1}}\big|_1^b = \frac{-1}{(p-1)b^{p-1}} + \frac{1}{p-1} \longrightarrow \frac{1}{p-1}$ as $b \longrightarrow +\infty$.

(v) $\int_0^1 \frac{dx}{x}$ diverges to $+\infty$: $\int_a^1 \frac{dx}{x} = \log x\big|_a^1 = -\log a \longrightarrow +\infty$ as $a \to 0^+$.

(vi) For any $0 < p < 1$, $\int_0^1 \frac{dx}{x^p}$ convegres to $\frac{1}{1-p}$: $\int_a^1 \frac{dx}{x^p} = \frac{-1}{(p-1)x^{p-1}}\big|_a^1 = \frac{-1}{p-1} + \frac{a^{1-p}}{p-1} \longrightarrow \frac{1}{1-p}$ as $a \to 0^+$. (For $p \leq 0$ the integral is proper.)

If $f \geq 0$ on (α, β) then the function $F(x) = \int_c^x f$ is increasing, so the improper integral $\int_\alpha^\beta f$ exists, but may be infinite; it converges iff it is $< \infty$. We therefore have the following so-called *comparison principle*:

Theorem 6.7.1. *Let f and g be functions locally integrable on a (possibly, unbounded) interval (α, β) such that $0 \leq f \leq g$. If $\int_\alpha^\beta g$ converges, then $\int_\alpha^\beta f$ also converges.*

Proof. Fix $c \in (\alpha, \beta)$, let $F(x) = \int_c^x f$ and $G(x) = \int_c^x f$ be integral functions of f and g. Then $F(x) \leq G(x)$ for all $x \in (c, \beta)$, so, if $\lim_{b\to\beta^-} G(b) < +\infty$, then also $\lim_{b\to\beta^-} F(b) < +\infty$. For all $x \in (\alpha, c)$ we have $F(x) \geq G(x)$ (since $F(x) = \int_c^x f = -\int_c^a f$). So, if $\lim_{a\to\alpha^+}(-G(a)) < +\infty$, then also $\lim_{a\to\alpha^+}(-F(a)) = +\infty$. ■

Examples. (i) If $p < 1$, then for any $x > 1$, $\frac{1}{x} < \frac{1}{x^p}$; since $\int_1^{+\infty} \frac{dx}{x} = +\infty$, $\int_1^{+\infty} \frac{dx}{x^p} = +\infty$ as well.

(ii) If $p > 1$, then for any $0 < x < 1$, $\frac{1}{x} < \frac{1}{x^p}$; since $\int_0^1 \frac{dx}{x} = +\infty$, $\int_0^1 \frac{dx}{x^p} = +\infty$ as well.

The *Gamma-function* Γ is defined as the improper integral $\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t} dt$, $x > 0$. This integral converges (and so, Γ is defined) for every $x > 0$. Indeed, $t^{x-1}e^{-t} < e^{-t/2}$ for all t large enough and $\int_1^{+\infty} e^{-t/2} < +\infty$, so $\int_1^{+\infty} t^{x-1}e^{-t} dt$ converges by the comparison principle. If $x \geq 1$, $\int_0^1 t^{x-1}e^{-t} dt$ is a proper integral. If $0 < x < 1$, $t^{x-1}e^{-t} \leq t^{x-1}$ for all $t \geq 0$ and $\int_0^1 t^{x-1} dt < +\infty$, so $\int_0^1 t^{x-1}e^{-t} dt$ converges by the comparison principle.

The Gamma-function satisfies the following functional equation:

Theorem 6.7.2. $\Gamma(1) = 1$, *and* $\Gamma(x+1) = x\Gamma(x)$ *for all* $x > 0$.

Proof. $\Gamma(1) = \int_0^{+\infty} e^{-t}dt = -e^{-t}\big|_0^{+\infty} = 1$. For any $x > 0$, integrating by parts we get

$$\int t^x e^{-t}dt = -\int t^x\, de^{-t} = -t^x e^{-t} + \int e^{-t}dt^x = -t^x e^{-t} + x\int e^{-t}t^{x-1}dt.$$

So,

$$\Gamma(x+1) = \int_0^{+\infty} t^x e^{-t}dt = \lim_{\substack{a\to 0^+\\ b\to+\infty}} (-t^x e^{-t})\big|_a^b + x\int_0^{+\infty} e^{-t}t^{x-1}dt = -\lim_{b\to+\infty} b^x e^{-b} + 0^x e^{-0} + x\Gamma(x) = x\Gamma(x).$$

■

It follows that $\Gamma(2) = 1$, $\Gamma(3) = 2$, $\Gamma(4) = 3\cdot 2 = 3!$, and by induction, $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$. Gamma-function is "a natural" extension of $n!$: $x!$ is defined for every $x > 0$ as $\Gamma(x+1)$. Using the equation $\Gamma(x) = \Gamma(x+1)/x$, the Gamma function can also be "naturally" extended to $(-1,0)$, then $(-2,-1)$, etc., that is, on $\mathbb{R}\setminus\{0,-1,-2,\ldots\}$.

We say that an improper integral $\int_\alpha^\beta f$ *converges absolutely* if $\int_\alpha^\beta |f| < \infty$.

Theorem 6.7.3. *If an improper integral converges absolutely then it converges.*

Proof. I'll consider the case where the integral is improper at β only. If $\int_a^\beta |f|$, improper at β, converges, then by the Cauchy's criterion for funcitons, Theorem 3.6.1, for any $\varepsilon > 0$ there is a (left) neighborhood U of β such that for any $x, y \in U$, with $x < y$, $\int_x^y |f| = \int_a^y |f| - \int_a^x |f| < \varepsilon$, and then

$$\Big|\int_a^y f - \int_a^x f\Big| = \Big|\int_x^y f\Big| \le \int_x^y |f| < \varepsilon.$$

So, by the same Cauchy criterion, $\int_a^\beta f$ converges. ■

Examples. (i) The integral $\int_1^{+\infty} x^{-2}\sin x$ converges. Indeed, $\big|x^{-2}\sin x\big| \le x^{-2}$ and $\int_1^{+\infty} x^{-2}$ converges, so $\int_1^{+\infty} x^{-2}\sin x$ converges absolutely.
(ii) The integral $\int_0^{+\infty}\sin(x^2)$ also converges. $\int_0^1 \sin(x^2)$ is proper, so we focus on $\int_1^{+\infty}\sin(x^2)$. We have

$$\int \sin x^2 dx = \tfrac12\int x^{-1}\sin x^2 d(x^2) = -\tfrac12\int x^{-1}d\cos(x^2) = -\tfrac12 x^{-1}\cos(x^2) + \tfrac12\int \cos(x^2)d(x^{-1})$$
$$= -\tfrac12 x^{-1}\cos(x^2) - \tfrac12\int x^{-2}\cos(x^2)dx.$$

So,

$$\int_1^{+\infty}\sin(x^2)dx = \lim_{b\to+\infty}\int_1^b \sin(x^2)dx = -\tfrac12\lim_{b\to+\infty} x^{-1}\cos(x^2)\big|_1^b - \tfrac12\int_1^{+\infty} x^{-2}\cos(x^2)dx.$$

Since $\lim_{b\to+\infty} b^{-1}\cos(b^2) = 0$ and $\int_1^{+\infty} x^{-2}\cos(x^2)dx$ converges absolutely, $\int_0^{+\infty}\sin(x^2)$ converges.

6.8. Arc length

A curve γ in the plane is a pair $(f(t), g(t))$ of continuous functions $[a,b] \longrightarrow \mathbb{R}$.

The arc length of γ is defined in the following way: For a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a,b]$, let Δs_i be the distance between the points $(f(t_{i-1}), g(t_{i-1}))$ and $(f(t_i), g(t_i))$, that is, $\Delta s_i = \sqrt{(f(t_i)-f(t_{i-1}))^2 + (g(t_i)-g(t_{i-1}))^2}$; define $L(\gamma, P) = \sum_{i=1}^n \Delta s_i$. Clearly, if P' is a refinement of P, then $L(\gamma, P') \ge L(\gamma, P)$. (When we add one point z in an interval $(t_{i-1}, t_i$, we replace Δs_i by the sum $\Delta s_i' + \Delta s_i''$ where $\Delta s_i'$ is the distance between $(f(t_{i-1}), g(t_{i-1}))$ and $(f(z), g(z))$, and $\Delta s_i''$ is the distance between $(f(t_i), g(t_i))$ and $(f(z), g(z))$; by the triangle inequality in the plane, $\Delta s_i' + \Delta s_i'' \ge \Delta s_i$.) We now define $L(\gamma) = \sup\big\{L(\gamma, P),\ P \text{ is a partition of } [a,b]\big\}$, and call it *the arc length of* γ; if $L(\gamma) < \infty$ we say that γ is *rectifiable*.

Differentiable curves with integrable derivatives are rectifiable. For simplicity, let's consider curves of the form $(x, f(x))$ only:

Theorem 6.8.1. *Let function f be continuous on an interval $[a,b]$, differentiable on (a,b), and such that f' is integrable, and let γ be the curve $(x,f(x))$, $x\in[a,b]$. Then γ is rectifiable with $L(\gamma)=\int_a^b\sqrt{1+f'(x)^2}\,dx$.*

Proof. Define $h(x)=\sqrt{1+f'(x)^2}$, $x\in[a,b]$; since f' is integrable, h is also integrable. Let $P=\{a=x_0<x_1<\cdots<x_n=b\}$ be a partition of $[a,b]$. For every i, by the M.V.T. there exists $z_i\in(x_{i-1},x_i)$ such that $f(x_i)-f(x_{i-1})=f'(z_i)\Delta x_i$, so that

$$\Delta s_i=\sqrt{(x_i-x_{i-1})^2+(f(x_i)-f(x_{i-1}))^2}=\sqrt{1+f'(z_i)^2}\Delta x_i=h(z_i)\Delta x_i.$$

Hence, $L(\gamma,P)=\sum_{i=1}^n\Delta s_i=S(h,P,\sigma)$ where σ is the selection $\{z_1,\ldots,z_n\}$.

Now let (P_n) be a sequence of partitions of $[a,b]$ such that $\Delta(h,P_n)\longrightarrow 0$ and $L(\gamma,P_n)\longrightarrow L(\gamma)$. (Such a sequence can be constructed by letting $P_n=P_n'+P_n''$ where $\Delta(h,P_n')\longrightarrow 0$ and $L(\gamma,P_n'')\longrightarrow L(\gamma)$.) For each n let σ_n be a selection subordinate to P_n such that $L(\gamma,P_n)=S(h,P_n,\sigma_n)$; as $S(h,P_n,\sigma_n)\longrightarrow\int_a^b h$, we obtain that $\int_a^b h=L(\gamma)$. ■

Example. Consider the curve $\gamma=(x,\sqrt{1-x^2})$, $x\in[-1,1]$, the upper arc of the circle $x^2+y^2=1$. The derivative $f'(x)=\frac{-x}{\sqrt{1-x^2}}$ of $f(x)=\sqrt{1-x^2}$ is not integrable (not bounded) on $[-1,1]$, but is locally integrable. For any $y>0$, the arc length of the curve $(x,f(x))$, $x\in[0,y]$, is $\int_0^y\sqrt{1+f'(x)^2}\,dx=\int_0^y\sqrt{1+\frac{x^2}{1-x^2}}dx=\int_0^y\frac{dx}{\sqrt{1-x^2}}$. This "arc length" function is called arcsin; we therefore have that arcsin is a primitive of $\frac{1}{\sqrt{1-x^2}}$, $\arcsin' x=\frac{1}{\sqrt{1-x^2}}$, with $\arcsin 0=0$.

The total length of γ is $\int_{-1}^1\frac{dx}{\sqrt{1-x^2}}$; this is an improper integral, which, actually, converges: since $\frac{1}{\sqrt{1-x^2}}\le\frac{1}{\sqrt{1-x}}$ on $[0,1]$ and (improper) $\int_0^1\frac{dx}{\sqrt{1-x}}=2$, $\int_0^1\frac{dx}{\sqrt{1-x^2}}$ converges by Theorem 6.7.1; and similarly $\int_{-1}^0\frac{dx}{\sqrt{1-x^2}}$ converges since $\frac{1}{\sqrt{1-x^2}}\le\frac{1}{\sqrt{1+x}}$ on $[-1,0]$ and $\int_{-1}^0\frac{dx}{\sqrt{1+x}}=2$. The number $\int_{-1}^1\frac{dx}{\sqrt{1-x^2}}$ is called π; we have $2\le\pi\le 4$. (Well, I cheated: we didn't prove an analogue of Theorem 6.8.1 for improper integrals. Ok, we can apply this theorem to intervals of the form $[-a,a]$ with $0<a<1$, and prove that $L(\gamma_a)\longrightarrow L(\gamma)$ as $a\to 1^-$, where $\gamma_a=\gamma|_{[-a,a]}$.)

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7. Taylor polynomials

7.1. Functions equal up to order n and Taylor polynomials

We say that two functions f and g are *equal up to order n at point a* if $f(x)-g(x)=o((x-a)^n)$ as $x\longrightarrow a$, that is, if $\frac{f(x)-g(x)}{(x-a)^n}\longrightarrow 0$ as $x\longrightarrow a$.

For two polynomials, a criterion of "being equal up to order n" is easy. For $a\in\mathbb{R}$ and $k,n\in\mathbb{N}$ we have $(x-a)^k=o((x-a)^n)$ as $x\longrightarrow a$ iff $k>n$. Given a polynomial $p(x)=a_0+a_1x+\cdots+a_kx^k$ and a point $a\in\mathbb{R}$ we may always rewrite p in terms of powers of $x-a$: $p(x)=a_0+a_1((x-a)+a)+\cdots+a_k((x-a)+a)^k$, and after opening the outer parentheses with the help of the binomial formula, we get $p(x)=b_0+b_1(x-a)+\cdots+b_k(x-a)^k$ for some $b_0,\ldots,b_k\in\mathbb{R}$. Given $n\in\mathbb{N}$, let's define *the truncation of p at a to degree n* $T_{a,n}p(x)=b_0+b_1(x-a)+\cdots+b_n(x-a)^n$. (If $k<n$, we put $b_{k+1}=\cdots=b_n=0$.) We have:

Theorem 7.1.1. *Two polynomials p and q are equal up to order n at a iff $T_{a,n}p=T_{a,n}q$.*

Proof. If $T_{a,n}p=T_{a,n}q$, then $p(x)-q(x)=c_{n+1}(x-a)^{n+1}+\cdots+c_k(x-a)^k$ for some $c_{n+1},\ldots,c_k$, so $p(x)-q(x)=o((x-a)^n)$ as $x\longrightarrow a$.

If $T_{a,n}p\ne T_{a,n}q$, then $p(x)-q(x)=c_d(x-a)^d+\cdots+c_k(x-a)^k$ for some $d\le n$ and $c_d,\ldots,c_k$ with $c_d\ne 0$. Then $\frac{p(x)-q(x)}{(x-a)^n}=\frac{1}{(x-a)^{n-d}}\big(c_d+c_{d+1}(x-a)+\cdots+c_k(x-a)^{k-n}\big)$. We have $\frac{1}{(x-a)^{n-d}}\longrightarrow\infty$ (if $n>d$) or $=1$ (if $n=d$), and $c_d+c_{d+1}(x-a)+\cdots+c_k(x-a)^{k-d}\longrightarrow c_d\ne 0$ as $x\longrightarrow a$. So, $\frac{p(x)-q(x)}{(x-a)^n}\not\longrightarrow 0$. ■

If f and g are n-times differentiable it is also easy to find out whether f and g are equal up to order n.

Theorem 7.1.2. *Let functions f and g be n-times differentiable at a with $f(a)=g(a)$, $f'(a)=g'(a)$, …, $f^{(n)}(a)=g^{(n)}(a)$. Then f and g are equal up to order n at a.*

Proof. Define $h = f - g$; then h is n-times differentiable at a with $h^{(k)}(a) = 0$ for all $k \leq n$, and we need to prove that $h(x) = o((x-a)^n)$ as $x \longrightarrow a$. "h is n-times differentiable at a" means that h is $(n-1)$-times differentiable in a neighborhood of a and $h^{(n-1)}$ is differentiable at a. Since h and $(x-a)^n$ are differentiable in a neighborhood of a, $\big((x-a)^n\big)' \neq 0$ for all $x \neq a$, and both $h(x), (x-a)^n \longrightarrow 0$ as $x \longrightarrow a$, by the L'Hospital's rule, $\lim_{x\to a} \frac{h(x)}{(x-a)^n} = 0$ if $\lim_{x\to a} \frac{h'(x)}{n(x-a)^{n-1}} = 0$. In its turn, this is so if $\lim_{x\to a} \frac{h''(x)}{n(n-1)(x-a)^{n-2}} = 0$, and by induction, if $\lim_{x\to a} \frac{h^{(n-1)}(x)}{n!(x-a)} = 0$. But since $h^{(n-1)}(a) = 0$, $\lim_{x\to a} \frac{h^{(n-1)}(x)}{x-a} = \lim_{x\to a} \frac{h^{(n-1)}(x)-h^{(n-1)}(a)}{x-a} = h^{(n)}(a) = 0$. ■

Let f be a function n-times differentiable at a point a. It follows from Theorem 7.1.2 that if we want to best approximate f at a by a polynomial p of degree $\leq n$, we need $p^{(k)}(a) = f^{(k)}(a)$ for all $k = 0, 1, \ldots, n$. Let $p(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots + a_n(x-a)^n$, then $p(a) = a_0$;
$p'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1}$, so $p'(a) = a_1$;
$p''(x) = 2a_2 + 3\cdot 2(x-a) + \cdots + n(n-1)(x-a)^{n-2}$, so $p''(a) = 2a_2$;
and by induction, $p^{(k)}(a) = k!a_k$ for all $k \leq n$.
So, if we want $p^{(k)}(a) = f^{(k)}(a)$ for all k, we need $a_k = \frac{f^{(k)}(a)}{k!}$.

The polynomial $P_{a,n,f}(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x-a)^k$ is called *the n-th Taylor polynomial* of f at a; it is characterized by the property $P_{a,n,f}^{(k)}(a) = f^{(k)}(a)$ for all $k = 0, 1, \ldots, n$. We therefore have the following fact, known as *Maclaurin's formula*:

Theorem 7.1.3. *Let f be a function n-times differentiable at a point a, and let $P_{a,n,f}$ be the n-th Taylor polynomial of f at a. Then $f(x) = P_{a,n,f}(x) + o((x-a)^n)$.*

The converse is true in the following form: if p is a polynomial such that $f(x) = p(x) + o((x-a)^n)$, then $T_{a,n}(p) = P_{a,n,f}$. Indeed, this is the case, p is equal to $P_{a,n,f}$ up to order n at a, so by Theorem 7.1.1, since $\deg P_{a,n,f} \leq n$, $T_{a,n}(p) = T_{a,n}(P_{a,n,f}) = P_{a,n,f}$.

7.2. Operations on Taylor polynomials

For a function f that is n-times differentiable at a point a, let $P_{a,n,f}$ be the n-th Taylor polynomial of f at a.

Theorem 7.2.1. *Let f be a function n-times differentiable at a point a. Then*
(i) $P_{a,n-1,f'} = P'_{a,n,f}$;
(ii) *if F is a primitive of f, then $P_{a,n+1,F}$ is the primitive of $P_{a,n,f}$ such that $P_{a,n+1,F}(a) = F(a)$;*
(iii) *for any $c \in \mathbb{R}$, $P_{a,n,cf} = cP_{a,n,f}$.*
(iv) *for any $d \in \mathbb{N}$, for $q(x) = f(x^d + a)$, $P_{0,dn,q}(x) = P_{a,n,f}(x^d + a)$.*
Let also g be a function n-times differentiable at a. Then
(v) $P_{a,n,f+g} = P_{a,n,f} + P_{a,n,g}$;
(vi) $P_{a,n,fg} = T_{a,n}(P_{a,n,f}P_{a,n,g})$;
(vii) *if $g(a) \neq 0$, then $P_{a,n,f/g}$ is the polynomial of degree $\leq n$ such that $T_{a,n}(P_{a,n,f/g}P_{a,n,g}) = P_{a,n,f}$.*
Let also h be a function n-times differentiable at $f(a)$. Then
(viii) $P_{a,n,h\circ f} = T_{a,n}(P_{f(a),n,h} \circ P_{a,n,f})$;
(ix) *if f is invertible in a neighborhood of a, then $P_{f(a),n,f^{-1}}$ is the polynomial of degree $\leq n$ such that $T_{a,n}\big(P_{f(a),n,f^{-1}} \circ P_{a,n,f}\big) = x$.*

Proof. Given a function u n-times differentiable at a, to prove that $T_{a,n}p = P_{a,n,u}$ for a polynomial p, it suffices to check that $p^{(k)}(a) = u^{(k)}(a)$ for all $k \leq n$, or that $p(x) = u(x) + o((x-a)^n)$.
(i) f' is $(n-1)$-times differentiable at a, so $P_{a,n-1,f'}$ is defined. Since for any $k \leq n-1$ we have $(P'_{a,n,f})^{(k)}(a) = P_{a,n,f}^{(k+1)}(a) = f^{(k+1)}(a) = (f')^{(k)}(a)$, we see that $P_{a,n-1,f'} = P'_{a,n,f}$.
(ii) follows from (i).
(iii) For any $k \leq n$, $(cP_{a,n,f})^{(k)}(a) = cP_{a,n,f}^{(k)}(a) = cf^{(k)}(a) = (cf)^{(k)}(a)$.
(iv) $q(x) = f(x^d + a) = P_{a,n,f}(x^d + a) + o((x^d + a - a)^n) = P_{a,n,f}(x^d + a) + o(x^{dn})$ as $x \longrightarrow 0$, so $P_{a,n,f}(x^d + a) = P_{0,dn,q}$.

(v) For any $k \le n$, $(P_{a,n,f} + P_{a,n,g})^{(k)}(a) = P^{(k)}_{a,n,f}(a) + P^{(k)}_{a,n,g}(a) = f^{(k)}(a) + g^{(k)}(a) = (f+g)^{(k)}(a)$.
(vi) We have $f(x) = P_{a,n,f}(x) + o((x-a)^n)$ and $g(x) = P_{a,n,g}(x) + o((x-a)^n)$, so

$$(fg)(x) = P_{a,n,f}(x)P_{a,n,g}(x) + P_{a,n,g}(x)o((x-a)^n) + P_{a,n,f}(x)o((x-a)^n) + o((x-a)^n)o((x-a)^n). \quad (7.1)$$

For any function u continuous at a (or just bounded in a neighborhood of a) we have $\lim_{x\to a} \frac{u(x)o((x-a)^n)}{(x-a)^n} = 0$, so $u(x)o((x-a)^n) = o((x-a)^n)$; thus (7.1) says that $(fg)(x) = P_{a,n,f}(x)P_{a,n,g}(x) + o((x-a)^n)$.
(vii) follows from (vi).
(viii) This item is less trivial. For simplicity, let's assume that $a = 0$ and $f(a) = 0$; the general case is obtained by replacing $f(x)$ by $f(a+x) - f(a)$ and $h(y)$ by $h(y + f(a))$. Let $P_{0,n,f}(x) = a_1x + \cdots + a_nx^n$ and $P_{0,n,h}(y) = b_0 + b_1y + \cdots + b_ny^n$, we have $f(x) = P_{0,n,f}(x) + o(x^n)$ and $h(y) = P_{0,n,h}(y) + \varphi(y)$ with $\lim_{y\to 0} \frac{\varphi(y)}{y^n} = 0$ and $\varphi(0) = 0$, so

$$h(f(x)) = P_{0,n,h}(f(x)) + \varphi(f(x)). \quad (7.2)$$

For any k, $f(x)^k = \big(P_{0,n,f}(x) + o(x^n)\big)^k = P_{0,n,f}(x)^k + o(x^n)$ since $o(x^n)\psi(x) = o(x^n)$ for any function ψ continuous at 0. Hence,

$$P_{0,n,h}(f(x)) = \sum_{k=0}^{n} b_k f(x)^k = \sum_{k=0}^{n} b_k\big(P_{0,n,f}(x)^k + o(x^n)\big) = P_{0,n,h}(P_{0,n,f}(x)) + o(x^n).$$

As for the second summand in (7.2), define $\eta(x) = \varphi(x)/x^n$ for $x \neq 0$ and $\eta(0) = 0$, then η is continuous at 0 and $\frac{\varphi(f(x))}{x^n} = \eta(f(x))\frac{f(x)^n}{x^n}$ for all x in a neighborhood of a, so

$$\lim_{x\to 0} \frac{\varphi(f(x))}{x^n} = \lim_{x\to 0} \eta(f(x)) \lim_{x\to 0}\Big(\frac{f(x)}{x}\Big)^n = 0 \cdot (f'(0))^n = 0.$$

So, $\varphi(f(x)) = o(x^n)$. Hence, by (7.2), $h(f(x)) = P_{0,n,h}(P_{0,n,f}(x)) + o(x^n)$.
(ix) follows from (viii). ■

This theorem reduces calculations of derivatives to operations with polynomials.

Examples. (i) Let f and g be n-times differentiable at a; we'll use Taylor polynomials to find a formula for $(fg)^{(n)}(a)$. Let $P_{a,n,f}(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$ and $P_{a,n,g}(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots + b_n(x-a)^n$. Then

$$\begin{aligned} &P_{a,n,fg}(x) = T_{a,n}\big(P_{a,n,f}(x)P_{a,n,g}(x)\big) \\ &= a_0b_0 + \big(a_0b_1 + a_1b_0\big)(x-a) + \big(a_0b_2 + a_1b_1 + a_2b_0\big)(x-a)^2 + \cdots + (a_0b_n + a_1b_{n-1} + \cdots + a_nb_0)(x-a)^n \\ &= \textstyle\sum_{k=0}^{n}\big(\sum_{i=0}^{k} a_ib_{k-i}\big)(x-a)^k. \end{aligned}$$

Hence,

$$\begin{aligned} (fg)^{(n)}(a) &= n!\sum_{i=0}^{n} a_ib_{n-i} = n!\sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}\cdot\frac{g^{(n-i)}(a)}{(n-i)!} = \sum_{i=0}^{n}\frac{n!}{i!(n-i)!}f^{(i)}(a)g^{(n-i)}(a) \\ &= \sum_{i=0}^{n}\tbinom{n}{i} f^{(i)}(a)g^{(n-i)}(a). \end{aligned}$$

(ii) Now, let's find the 3-rd Taylor polynomial $P_{0,3,f}$ for $f(x) = \tan x = \frac{\sin x}{\cos x}$. We must have $T_{0,3}\big(P_{0,3,f}P_{0,3,\cos}\big) = P_{0,3,\sin}$, so if $P_{3,0,f}(x) = c_0 + c_1x + c_2x^2 + c_3x^3$, then $T_{0,3}\big(c_0 + c_1x + c_2x^2 + c_3x^3\big)\big(1 - \frac{x^2}{2}\big) = x - \frac{x^3}{6}$. This means that $c_0 = 0$, $c_1 = 1$, $c_2 - c_0\frac{1}{2} = 0$ so $c_2 = 0$, $c_3 - c_1\frac{1}{2} = -\frac{1}{6}$ so $c_3 = \frac{1}{3}$. So, $P_{0,3,f}(x) = x + \frac{x^3}{3}$. It now follows that $\tan'(0) = 1$, $\tan''(0) = 0$, $\tan'''(0) = 3!\frac{1}{3} = 2$.

7.3. Taylor polynomials of basic functions

Let's find the Taylor polynomials at 0 of basic functions.

(i) Let $f(x) = e^x$. Then for any k, $f^{(k)}(x) = e^x$, so $f^{(k)}(0) = 1$, so for any n,

$$P_{0,n,f}(x) = 1 + 1x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

(ii) Let $f(x) = \sin x$. Then $f'(x) = \cos x$, $f''(x) = -\sin x$, $f^{(3)}(x) = -\cos x$, $f^{(4)}(x) = \sin x$, ..., so $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f^{(3)}(0) = -1$, $f^{(4)}(0) = 0$, ..., so for any n,

$$P_{0,2n+1,f}(x) = P_{0,2n+2,f}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{k=0}^{n}(-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

(iii) Let $f(x) = \cos x$. Then $f = \sin' x$, so for any n, $P_{0,2n,f}(x) = P'_{0,2n+1,\sin}(x)$, that is,

$$P_{0,2n,f}(x) = P_{0,2n+1,f}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{k=0}^{n}(-1)^k \frac{x^{2k}}{(2k)!}.$$

(iv) Let $f(x) = \frac{1}{1-x}$, then for any n, $f(x) = \frac{1-x^{n+1}}{1-x} + \frac{x^{n+1}}{1-x} = 1 + x + x^2 + \cdots + x^n + o(x^n)$, so

$$P_{0,n,f}(x) = 1 + x + x^2 + \ldots + x^n = \sum_{k=0}^{n} x^k.$$

(v) Let $f(x) = \frac{1}{1+x}$, then $f(x) = g(-x)$ where $g(x) = \frac{1}{1-x}$, so for any n,

$$P_{0,n,f}(x) = P_{0,n,g}(-x) = 1 - x + x^2 - \ldots + (-1)^n x^n = \sum_{k=0}^{n}(-1)^k x^k.$$

(vi) Let $f(x) = \log(1+x)$, then f is a primitive of $g(x) = \frac{1}{1+x}$,

$$P_{0,n,f}(x) = \int P_{0,n-1,g}(x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} = C + \sum_{k=1}^{n}(-1)^{k-1}\frac{x^k}{k},$$

where $C = f(0) = 0$.

(vii) Let $f(x) = \frac{1}{1+x^2}$, then $f(x) = g(x^2)$ where $g(x) = \frac{1}{1+x}$, so for any n,

$$P_{0,2n,f}(x) = T_{0,2n}(P_{0,2n,g}(x^2)) = 1 - x^2 + x^4 - \ldots + (-1)^n x^{2n} = \sum_{k=0}^{n}(-1)^k x^{2k}.$$

(viii) Let $f(x) = \arctan x = \tan^{-1} x$, then f is a primitive of $g(x) = \frac{1}{1+x^2}$, so for any n,

$$P_{0,2n+1,f}(x) = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} = C + \sum_{k=0}^{n}(-1)^k \frac{x^{2k+1}}{2k+1},$$

where $C = \arctan 0 = 0$.

(ix) Let $\alpha \in \mathbb{R}$ and $f(x) = (1+x)^\alpha$. Then $f'(x) = \alpha(1+x)^{\alpha-1}$, $f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$, ..., so $f(0) = 1$, $f'(0) = \alpha$, $f''(0) = \alpha(\alpha-1)$, ..., for any n, $f^{(n)}(0) = \alpha(\alpha-1)\cdots(\alpha-n+1)$, so

$$P_{0,n,f}(x) = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n = \sum_{k=0}^{n}\binom{\alpha}{k}x^k,$$

where we define $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$.

7.4. Some applications of Taylor polynomials

The Taylor-Maclaurin formula can be used to calculate indeterminate limits:

Example. $\lim\limits_{x\to 0}\dfrac{\cos x - 1}{x^2} = \lim\limits_{x\to 0}\dfrac{1 - x^2/2 + o(x^2) - 1}{x^2} = \lim\limits_{x\to 0}\dfrac{-x^2/2 + o(x^2)}{x^2} = \dfrac{-1}{2}.$

If the Taylor polynomial of f at a is somehow known, we can use it to find the derivatives of f at a:

Example. Since, for any n, the $2n$-th Taylor polynomial at 0 of $f(x) = e^{x^2}$ is $\sum_{k=0}^{n}\frac{x^{2k}}{k!}$, for any k, $f^{(2k)}(0) = (2k)!/k!$ and $f^{(2k+1)}(0) = 0$.

The local behavior of a function is "the same" as the local behavior of its Taylor polynomial:

Theorem 7.4.1. *Let f be n times differentiable at a, let $f'(a) = f''(a) = \ldots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$. Then*
(i) *if n is even and $f^{(n)}(a) > 0$, f has a strict local minimum at a;*
(ii) *if n is even and $f^{(n)}(a) < 0$, f has a strict local maximum at a;*
(iii) *if n is odd and $f^{(n)}(a) > 0$, f is strictly increasing at a;*
(iv) *if n is odd and $f^{(n)}(a) < 0$, f is strictly decreasing at a.*

Proof. By Maclaurin's formula, $f(x) = f(a) + c(x-a)^n + \varphi(x)$ where $c = f^{(n)}(a)/n! \neq 0$ and $\varphi(x) = o((x-a)^n)$. In a neighborhood I of a, $|\varphi(x)| < \left|c(x-a)^n\right|$, so $c(x-a)^n + \varphi(x) > 0$ if $c(x-a)^n > 0$ and $c(x-a)^n + \varphi(x) < 0$ if $c(x-a)^n < 0$.

If n is even and $c > 0$, then $c(x-a)^n > 0$ for all $x \neq a$, so $f(x) > f(a)$ for all $x \in I \setminus \{a\}$, so f has a strict local minimum at a.

If n is even and $c < 0$, then $c(x-a)^n < 0$ for all $x \neq a$, so $f(x) < f(a)$ for all $x \in I \setminus \{a\}$, so f has a strict local maximum at a.

If n is odd and $c > 0$, then $c(x-a)^n > 0$ for all $x > a$ and $c(x-a)^n < 0$ for all $x < a$, so, $f(x) > f(a)$ for all $x \in I$ such that $x > a$ and $f(x) < f(a)$ for all $x \in I$ such that $x < a$, so f is strictly increasing at a.

If n is odd and $c < 0$, then $c(x-a)^n > 0$ for all $x < a$ and $c(x-a)^n < 0$ for all $x > a$, so, $f(x) > f(a)$ for all $x \in I$ such that $x < a$ and $f(x) < f(a)$ for all $x \in I$ such that $x > a$. so f is strictly decreasing at a. ■

7.5. Remainder in Taylor's formula

The Taylor polynomial of a function f at a point a "well approximates" f in a neighborhood a, but the Maclaurin formula gives no information about how large this neighborhood is. So, it is useless for calculating the values of the function: given a specific point x close to a, we cannot estimate the difference $f(x) - P_{a,n,f}(x)$, and cannot even say whether $P_{a,n,f}(x) \longrightarrow f(x)$ as $n \longrightarrow \infty$.

Example. Let $f(x) = e^{-1/|x|}$ as $x \neq 0$ and $f(0) = 0$; then $f^{(n)}(0) = 0$ for all n, so all Taylor polynomls of f at 0 are equal to 0. Hence, $P_{0,n,f} \not\longrightarrow f(x)$ as $n \longrightarrow \infty$ for all $x \neq 0$.

The function $R_{a,n,f} = f - P_{n.a.f}$ is called *the remainder* (in Taylor's formula); we have $f = P_{a,n,f} + R_{a,n,f}$, $R_{a,n,f}(a) = R'_{a,n,f}(a) = \cdots = R^{(n)}_{a,n,f}(a) = 0$, and $R_{a,n,f}(x) = o((x-a)^n)$. If f is $(n+1)$-times differentiable in a neighborhood of a then $R^{(n+1)}_{a,n,f}(x) = f^{(n+1)}(x)$ in this neighborhood.

Theorem 7.5.1. *Let the function $f\colon [a,b] \longrightarrow \mathbb{R}$ be n-times differentiable on $[a,b]$, $(n+1)$-times differentiable on (a,b), and $f^{(n)}$ be continuous at a and b. Then*

(i) *there exists $c \in (a,b)$ such that $R_{a,n,f}(b) = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$* (which is called "Lagrange's form of the remainder")*;*

(ii) *there exists $c \in (a,b)$ such that $R_{a,n,f}(b) = \frac{f^{(n+1)}(c)}{n!}(b-c)^n(b-a)$* (which is called "Cauchy's form of the remainder")*;*

(iii) *if, additionally, $f^{(n+1)}$ is integrable on $[a,b]$ then $R_{a,n,f}(b) = \int_a^b \frac{f^{(n+1)}(x)}{n!}(b-x)^n dx$* (which is called "the integral form of the remainder")*.*

All this remains true if f is not n-times differentiable at a and b but only n-times right-hand differentiable at a and n-times left-hand differentiable at b. Similar statements (with a and b switched) are true for the (left-hand) Taylor polynomial and remainder of f at b.

The Taylor formula with remainder in Lagrange's form is a generalization of Lagrange's M.V.T. Notice also that if $f^{(n+1)}$ is continuous on $[a,b]$, then (i) and (ii) follow from (iii) by M.V.T.-s for integrals.

It follows that if f is $(n+1)$-times differentiable on an interval I and $a \in I$ then
(i) for every $x \in I$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some $c = c_x \in [a,x]$ or $[x,a]$;
(ii) for every $x \in I$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a)$$

for some $c = c_x \in [a,x]$ or $[x,a]$;
(iii) if, additionally, $f^{(n+1)}$ is locally integrable on I, then for every $x \in I$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt.$$

Proof. (i) Put $R = R_{a,n,f}$, then $R^{(n)}$ is continuous on $[a,b]$ and $R(a) = R'(a) = \cdots = R^{(n)}(a) = 0$. By Cauchy's M.V.T. there exists $c_1 \in (a,b)$ such that

$$\frac{R(b)}{(b-a)^{n+1}} = \frac{R(b)-R(a)}{(b-a)^{n+1}} = \frac{R'(c_1)}{(n+1)(c_1-a)^n}.$$

Next, there exists $c_2 \in (a,c_1) \subseteq (a,b)$ such that

$$\frac{R'(c_1)}{(c_1-a)^n} = \frac{R'(c_1)-R'(a)}{(c_1-a)^n} = \frac{R''(c_2)}{n(c_2-a)^{n-1}},$$

so that

$$\frac{R(b)}{(b-a)^{n+1}} = \frac{R''(c_2)}{(n+1)n(c_2-a)^{n-1}}.$$

And so on (or by induction), we get that $\frac{R(b)}{(b-a)^{n+1}} = \frac{R^{(n+1)}(c)}{(n+1)!}$ for some $c = c_{n+1} \in (a,b)$.
(ii) The proof is more tricky. For every $x \in [a,b]$ we write the Taylor formula at x, $f(b) = P_{x,n,f}(b) + R_{x,n,f}(b)$, and define $p(x) = P_{x,n,f}(b)$ and $r(x) = R_{x,n,f}(b)$, $x \in [a,b]$; then p and r are continuous at a and b. For all $x \in (a,b)$ we have

$$\begin{aligned} p'(x) &= \Big(f(x) + f'(x)(b-x) + \tfrac{f''(x)}{2}(b-x)^2 + \cdots + \tfrac{f^{(n)}(x)}{n!}(b-x)^n\Big)' \\ &= f'(x) + f''(x)(b-x) - f'(x) + \tfrac{f^{(3)}(x)}{2}(b-x)^2 - f''(x)(b-x) + \cdots + \tfrac{f^{(n+1)}(x)}{n!}(b-x)^n - \tfrac{f^{(n)}(x)}{(n-1)!}(b-x)^{n-1} \\ &= \tfrac{f^{(n+1)}(x)}{n!}(b-x)^n, \end{aligned}$$

so $r'(x) = \big(f(b) - p(x)\big)' = -\frac{f^{(n+1)}(x)}{n!}(b-x)^n$. We also have $r(a) = R_{a,n,f}(b)$ and $r(b) = R_{b,n,f}(b) = 0$. By the M.V.T. there is $c \in (a,b)$ such that $r(b) - r(a) = r'(c)(b-a)$, so $R_{a,n,f}(b) = \frac{f^{(n+1)}(c)}{n!}(b-c)^n(b-a)$.

(iii) Let $R = R_{a,n,f} = f - P_{a,n,f}$. Since $P_{a,n,f}$ is a polynomial of degree $\leq n$, we have $P_{a,n,f}^{(n+1)} = 0$, so $f^{(n+1)} = R^{(n+1)}$. Since $R^{(n)}(a) = 0$, integrating by parts we obtain

$$\int_a^b \frac{R^{(n+1)}(x)}{n!}(b-x)^n dx = \frac{1}{n!}\int_a^b (b-x)^n dR^{(n)}(x) = \frac{1}{n!}(b-x)^n R^{(n)}(x)\big|_a^b - \frac{1}{n!}\int_a^b R^{(n)}(x)(-n)(b-x)^{n-1}dx$$
$$= -\frac{1}{n!}(b-a)^n R^{(n)}(a) + \frac{1}{(n-1)!}\int_a^b R^{(n)}(x)(b-x)^{n-1}dx = \int_a^b \frac{R^{(n)}(x)}{(n-1)!}(b-x)^{n-1}dx.$$

By induction, $\int_a^b \frac{R^{(n)}(x)}{(n-1)!}(b-x)^{n-1}dx = \int_a^b R'(x)\,dx = R(b) - R(a) = R(b)$. ■

The "Taylor formulas with remainder" allow to estimate the difference between a function and its Taylor polynomial.

Examples. (i) For any $x \in \mathbb{R}$, for any n, $\sin x = P_{0,2n+1,\sin}(x) + R_{0,2n+1,\sin}(x)$ where, in Lagrange's form, $R_{0,2n+1,\sin}(x) = \frac{h(c_n)}{(2n+2)!}x^{2n+2}$ where $c_n \in (0,x)$, $h = \pm\sin$ or $\pm\cos$. Since $|h(c_n)| \leq 1$, $|R_{0.2n+1,\sin}(x)| \leq \frac{|x|^{2n+2}}{(2n+2)!}$. We see that for any x, $R_{0,2n+1,\sin}(x) \longrightarrow 0$ as $n \longrightarrow \infty$, that is, $P_{0,2n+1,\sin}(x) \longrightarrow \sin x$.

(ii) In the same way, for any x and n, $|R_{0,2n,\cos}(x)| \leq \frac{|x|^{2n+1}}{(2n+1)!}$ and $P_{0,2n+1,\cos}(x) \longrightarrow \cos x$ as $n \to \infty$.

(iii) For $\exp x = e^x$, for any x and any n, $R_{0,n,\exp}(x) = \frac{e^{c_n}}{(n+1)!}x^{n+1}$ for some $c_n \in (0,x)$. If $x > 0$, $0 < e^{c_n} < e^x$, if $x < 0$, $0 < e^c < 1$. So, in any case, $R_{0,n,\exp}(x) \longrightarrow 0$.

(iv) For the function $f(x) = \frac{1}{1+x}$ we have $f(x) = 1 - x + x^2 - \cdots \pm x^n \mp \frac{x^{n+1}}{1+x}$, so $R_{0,n,f}(x) = \mp\frac{x^{n+1}}{1+x}$ and no sophisticated formula is needed. As $n \longrightarrow \infty$, $R_{0,n,f}(x) \longrightarrow 0$ if $|x| < 1$ and $\not\longrightarrow 0$ if $|x| \geq 1$. This means that $P_{0,n,f}(x) \longrightarrow f(x)$ for all $x \in (-1,1)$ and diverges for all other x.

(v) For $f(x) = \log(1+x)$, $x > -1$, for any n, $f^{(n)}(x) = \pm\frac{(n-1)!}{(1+x)^n}$. Lagrange's form of the remainder is $R_{0,n,f}(x) = \pm\frac{n!}{(n+1)!(1+c_n)^n}x^{n+1} = \pm\frac{x^{n+1}}{(n+1)(1+c_n)^n}$ for some $c_n \in (0,x)$. If $0 < x \leq 1$, $|R_{0,n,f}(x)| < \frac{x^{n+1}}{n+1} \longrightarrow 0$ as $n \longrightarrow \infty$. For $-1 < x < 0$, $|x|$ can be $\geq |1+c_n|$, so this formula doesn't work. However, using Cauchy's form of the remainder, $R_{0,n,f} = \pm\frac{(x-c_n)^{n+1}}{(1+c_n)^n}x$ for some $c_n \in (x,0)$, since $\left|\frac{1}{1+c_n}\right| < \frac{1}{|1+x|}$ and $\frac{|x-c_n|}{|1+c_n|} < |x|$ (indeed, this is equivalent to $c_n - x < (-x)(1+c_n) = -x - xc_n$), we see that $|R_{0,n,f}(x)| < \frac{|x|^{n+1}}{|1+x|} \longrightarrow 0$ as well. So, $P_{0,n,f}(x) \longrightarrow \log(1+x)$ for all $x \in (-1,1]$. (We will learn later that, indeed, $P_{0,n,f}(x) \not\longrightarrow \log(1+x)$ for all $x > 1$.)

Another approach is to integrate the remainder of $\log(1+x)' = \frac{1}{1+x}$. For any $x > -1$, for any n, $R_{0,n,f}(x) = \int_0^x Q(t)\,dt$ where Q is the remainder in the Taylor formula of degree $(n-1)$ at 0 for $\frac{1}{1+x}$, $Q(x) = \frac{\pm x^n}{1+x}$.

(vi) Let $f(x) = \arctan x$, for which $f'(x) = \frac{1}{1+x^2}$. Let $n \in \mathbb{N}$. We have $\frac{1}{1+x} = 1 - x + x^2 - \cdots \pm x^n + Q(x)$ where $Q(x) = \frac{\pm x^{n+1}}{1+x}$. For $x > 0$, $|Q(x)| < x^{n+1}$. So, for any x, $\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots \pm x^{2n} + Q(x^2)$ with $|Q(x^2)| \leq x^{2n+2}$. Integrating this formula (and taking into account that $\arctan 0 = 0$) we get, for any $x > 0$, $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + \frac{(-1)^n x^{2n+1}}{2n+1} + R_{0.2n+1,f}(x)$, where $R_{0,2n+1,f}(x) = \int_0^x Q(t^2)dt$ and so $|R_{0,2n+1,f}(x)| \leq \int_0^x t^{2n+2}dt = \frac{|x|^{2n+3}}{2n+3}$. It follows that $R_{0,2n+1,f}(x) \longrightarrow 0$ and $P_{0,2n+1,f}(x) \longrightarrow f(x)$ for all $x \in [-1,1]$.

(vii) Let $\alpha \in \mathbb{R}$, $f(x) = (1+x)^\alpha$, $x > -1$. For any $n \in \mathbb{N}$, $f^{(n+1)}(x) = \alpha(\alpha-1)\cdots(\alpha-n)(1+x)^{\alpha-n-1}$. Lagrange's form of the remainder is $R_{0,n,f}(x) = \binom{\alpha}{n+1}(1+c_n)^{\alpha-n-1}x^{n+1}$ for some $c_n \in (0,x)$; Cauchy's form is $R_{0,n,f}(x) = \binom{\alpha}{n}(\alpha-n)(1+c_n)^{\alpha-n-1}(x-c_n)^n x$ for some $c_n \in (0,x)$. Lagrange's form shows that $R_{0,n,f}(x) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $x \in [0,1)$; Cauchy's form shows that this is so for all $x \in (-1,0)$; for all x with $|x| > 1$, $R_{0,n,f}(x) \not\longrightarrow 0$. (These facts are not easy to prove and we have to postpone this.)

For Euler's number e, for every $n \in \mathbb{N}$ we have $e = e^1 = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + r$ where $0 < r < \frac{e}{(n+1)!}$. We also know that $e < 3$, so $r < \frac{3}{(n+1)!}$. This allows to effectively calculate the value of e. We can also use this formula to prove:

Theorem 7.5.2. *e is irrational.*

Proof. Assume $e = \frac{m}{n}$ for $m, n \in \mathbb{N}$, $n \geq 3$, then $(n!)e \in \mathbb{N}$. But $n!r = n!\frac{3}{(n+1)!} = \frac{3}{n+1} < 1$, so $n!\big(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + r\big) = n! + n! + \frac{n!}{2!} + \cdots + \frac{n!}{n!} + n!r$ cannot be integer. ■

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8. Series

8.1. Convergent and divergent series

A series is a formal infinite sum $a_1 + a_2 + a_3 + \ldots$, or $\sum_{i=1}^{\infty} a_i$. *The n-th partial sum of* $\sum_{i=1}^{\infty} a_i$ is $s_n = \sum_{i=1}^{n} a_i$. If the sequence of partial sums (s_n) converges to s, we say that the series $\sum a_i$ *converges* and write $s = \sum_{i=1}^{\infty} a_i$. If $\lim s_n = \infty$, we say that the series $\sum a_i$ *diverges to* ∞ and write $\sum a_i = \infty$. If (s_n) diverges, we say that $\sum a_i$ *diverges*.

Notice that convergence or divergence of a series doesn't depend on finitely many its terms: $\sum_{i=1}^{\infty} a_i$ converges iff $\sum_{i=k}^{\infty} a_i$ converges for any $k \in \mathbb{N}$. (But, of course, the sum of the series depends on every its term.)

Examples.

(i) Partial sums of *the geometric series* $\sum a^i$ are $\frac{1-a^{n+1}}{1-a}$, $n \in \mathbb{N}$. Thus the series converges iff $|a| < 1$, in which case $\sum_{i=1}^{\infty} a^i = \frac{1}{1-a}$; diverges to $+\infty$ if $a \geq 1$; diverges to ∞ if $a < -1$; and just diverges if $a = -1$.
(ii) Given a sequence (b_n), the corresponding *telescopic series* is $\sum_{i=1}^{\infty}(b_{i+1} - b_i)$; for any n, $\sum_{i=1}^{n}(b_{i+1} - b_i) = b_{n+1} - b_1$, so the series converges iff the sequence (b_n) converges. An example is $\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \sum_{i=1}^{\infty}\big(\frac{1}{i} - \frac{1}{i+1}\big) = 1$.
(iii) $\sum_{i=0}^{\infty} \frac{1}{i!} = 1 + 1 + \frac{1}{2!} + \cdots = e$, since, as we know, $e^1 - \sum_{i=0}^{n} \frac{1}{i!} = R_{0,n,\exp}(x) \longrightarrow 0$ as $n \longrightarrow \infty$.

"The theory of series" is, actually, just another language for "the theory of sequences". Indeed, given a sequence (s_n), we can construct a series for which (s_n) is the sequence of partial sums by puting $a_1 = s_1$ and $a_i = s_i - s_{i-1}$ for all $i \geq 2$. Thus sequences and series are in one-to-one correspondence, and corresponding sequences and series converge or diverge simultaneously, with the sum of the series equal to the limit of the sequence. Therefore, all facts about convergence of sequences can be translated into the language of series, and vice versa.

Theorem 8.1.1. *If $\sum a_i$ and $\sum b_i$ converge, then $\sum(a_i + b_i)$ converges and $\sum_{i=1}^{\infty}(a_i + b_i) = \sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i$. If $\sum a_i$ converges, then $\sum(ca_i)$ converges for any $c \in \mathbb{R}$ and $\sum_{i=1}^{\infty}(ca_i) = c\sum_{i=1}^{\infty} a_i$.*

Proof. Let $s_n = \sum_{i=1}^{n} a_i$ and $r_n = \sum_{i=1}^{n} b_i$, $n \in \mathbb{N}$. Then for any n, $\sum_{i=1}^{n}(a_i + b_i) = s_n + r_n$ and $\sum_{i=1}^{n} ca_i = cs_n$. So, if the sequences (s_n) and (r_n) converge, the sequences of partial sums of $\sum(a_i + b_i)$ and $\sum(ca_i)$ also converge. ■

Theorems about convergence of series are often called *tests*. The simplest such test is *the vanishing test*:

Theorem 8.1.2. *If $\sum a_i$ converges, then* $\lim a_i = 0$.

Proof. Let $s_n = \sum_{i=1}^{n} a_i$, $n \in \mathbb{N}$. Since the sequence (s_n) converges, the sequence $a_n = s_n - s_{n-1}$ converges to $\lim s_n - \lim s_{n-1} = 0$. ■

Note that the condition $\lim a_i = 0$ is not sufficient for convergence of a series; for example, the series $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots$ and $1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \cdots$ diverge.

The necessary and sufficient condition for convergence of series is, of course, *the Cauchy criterion*:

Theorem 8.1.3. *A series $\sum a_i$ converges if and only if for any $\varepsilon > 0$ there exists k such that for any $n > m \geq k$, $\big|\sum_{i=m+1}^{n} a_i\big| < \varepsilon$.*

Proof. Let $s_n = \sum_{i=1}^{n} a_i$, $n \in \mathbb{N}$; then for any $n > m$, $s_n - s_m = \sum_{i=m+1}^{n} a_i$. And by Cauchy criterion for sequences, (s_n) converges iff for any $\varepsilon > 0$ there is k such that for any $n > m \geq k$, $|s_n - s_m| < \varepsilon$. ■

We say that a series $\sum a_i$ *converges absolutely* if the series $\sum |a_i|$ converges.

From Cauchy's criterion, we have:

Theorem 8.1.4. *If a series converges absolutely then it converges.*

Proof. If $\sum a_i$ converges absolutely then by the Cauchy's criterion, for any $\varepsilon > 0$ there is k such that for any $n > m \geq k$, $\sum_{i=m+1}^{n} |a_i| < \varepsilon$. But then $\left|\sum_{i=m+1}^{n} a_i\right| \leq \sum_{i=m+1}^{n} |a_i| < \varepsilon$, so $\sum a_i$ converges by the Cauchy criterion. ■

8.2. Convergence of series with nonnegative terms

We will list several tests for convergence of series with all nonnegative terms; these tests actually apply to any series in order to determine whether it converges absolutely.

For a series $\sum a_i$ with all $a_i \geq 0$, the sequence of partial sums is increasing. We therefore have:

Theorem 8.2.1. *If $a_i \geq 0$ for all $i \in \mathbb{N}$, then the series $\sum a_i$ converges iff the sequence $s_n = \sum_{i=1}^{n} a_i$ is bounded, and diverges to $+\infty$ otherwise.*

So, for a series $\sum a_i$ with $a_i \geq 0$ for all i, the statements "the series converges" and "$\sum a_i < +\infty$" are equivalent.

For series with nonnegative terms there are several easy but powerful "tests" for convergence. The most evident is *the comparison test*:

Theorem 8.2.2. *If $0 \leq a_i \leq b_i$ for all i and $\sum b_i < +\infty$, then $\sum a_i < +\infty$.*

Of course, "for all i" can be replaced by "for all i large enough".

Proof. If the sequence of partial sums of $\sum b_i$ is bounded, then the sequence of partial sums of $\sum a_i$ is bounded. ■

As a corollary, we have the following test:

Theorem 8.2.3. *If $a_i, b_i \geq 0$ for all i, $\sum a_i < +\infty$ and the sequence (b_i) is bounded, then $\sum a_i b_i < +\infty$.*

Proof. Let M be such that $b_i \leq M$ for all i; then $a_i b_i \leq M a_i$ for all i and $\sum (M a_i)$ converges, so $\sum a_i b_i$ converges by comparison test. ■

It follows that for any integer $d \geq 2$ and any sequence of "digits" $e_1, e_2, \ldots \in \{0, \ldots, d-1\}$ the series $e_1 d^{-1} + e_2 d^{-2} + \cdots = \sum_{i=1}^{\infty} e_i d^{-i}$ converges, namely to $x = .e_1 e_2 \ldots$ in the base d numerical system (after replacing each digit by the corresponding symbol).

The limit comparison test:

Theorem 8.2.4. *If $a_i, b_i > 0$ for all i and a finite nonzero $\lim(a_i/b_i)$ exists, then $\sum a_i < +\infty$ iff $\sum b_i < +\infty$.*

Proof. If $\lim(a_i/b_i) = c > 0$, then $a_i < 2cb_i$ and $b_i < \frac{2}{c} a_i$ for all i large enough, so by the comparison test, $\sum a_i < +\infty$ iff $\sum b_i < +\infty$. ■

The following two tests are based on comparison with a geometric series. *The root test*:

Theorem 8.2.5. *Let $a_i \geq 0$ for all i. If $\limsup \sqrt[i]{a_i} < 1$, then $\sum a_i < \infty$; if $\limsup \sqrt[i]{a_i} > 1$, then $\sum a_i = \infty$.*

Proof. Let $\limsup \sqrt[i]{a_i} < 1$. Take any c such that $\limsup \sqrt[i]{a_i} < c < 1$; then $\sqrt[i]{a_i} < c$ for all i large enough, so $a_i < c^i$ for all i large enough. Since $\sum c^i$ converges, so does $\sum a_i$.

Let now $\limsup \sqrt[i]{a_i} > 1$. Then $\sqrt[i]{a_i} > 1$ for infinitely many i, so $a_i > 1$ for such i, so $\sum a_i$ diverges by the vanishing test. ■

The ratio test:

Theorem 8.2.6. *Let $a_i > 0$ for all i. If $\limsup(a_{i+1}/a_i) < 1$, then $\sum a_i < +\infty$; if $\liminf(a_{i+1}/a_i) > 1$, then $\sum a_i = +\infty$.*

Proof. Let $\limsup(a_{i+1}/a_i) < 1$. Take any c such that $\limsup(a_{i+1}/a_i) < c < 1$, and find k such that $a_{i+1}/a_i < c$ for all $i \geq k$. Then $a_{k+1} < ca_k$, $a_{k+2} < ca_{k+1} < c^2 a_k$, and by induction, $a_i < c^{i-k} a_k = c^i (a_k/c^k)$ for all $i \geq k$. Since $\sum c^i$ converges, so does $\sum a_i$.

Let now $\liminf(a_{i+1}/a_i) > 1$. Then there is k such that $a_{i+1}/a_i \geq 1$ and so, $a_{i+1} \geq a_i$ for all $i \geq k$. So, $a_i \geq a_k > 0$ for all $i \geq k$, and $\sum a_i$ diverges by the vanishing test. ■

For series with decreasing non-negative terms there are two more nice tests. *The integral test*:

Theorem 8.2.7. *Let $f\colon [1,+\infty) \longrightarrow \mathbb{R}$ be a positive decreasing function and let $a_i = f(i)$, $i \in \mathbb{N}$. Then the series $\sum a_i$ converges iff the improper integral $\int_1^\infty f$ converges.*

Proof. Let $F(x) = \int_1^x f$, $x > 1$. Since $f > 0$, F is increasing, so $\int_1^{+\infty} f = \lim_{x\to+\infty} F(x)$ exists, finite or infinite, and is equal to $\lim_{n\to\infty} F(n)$ of the sequence $(F(n))$.

Put $b_i = \int_i^{i+1} f$, $i \in \mathbb{N}$, then for any n, $F(n) = \sum_{i=1}^{n-1} \int_i^{i+1} f = \sum_{i=1}^{n-1} b_i$; so $\int_1^{+\infty} f$ converges iff the series $\sum b_i$ converges. Since f is decreasing, for any i, $f(i) \geq f(x) \geq f(i+1)$ for all $x \in [i, i+1]$, so $a_i = f(i) \cdot 1 \geq b_i \geq f(i+1) \cdot 1 = a_{i+1}$. By comparison, we see that $\sum b_i$ converges iff $\sum a_i$ does. ■

The condensation test:

Theorem 8.2.8. *Let (a_i) be a nonnegative decreasing sequence. Then the series $\sum a_i$ converges iff the series $\sum 2^k a_{2^k}$ does.*

(To make it clear: $\sum_{k=1}^\infty 2^k a_{2^k} = 2a_2 + 4a_4 + 8a_8 + \cdots = (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + (a_8 + \cdots + a_8) + \cdots$.)

Proof. Let $s_m = \sum_{i=1}^m a_i$ and $r_n = \sum_{k=1}^n 2^k a_{2^k}$, $m, n \in \mathbb{N}$. For any n,

$$s_{2^n-1} = a_1 + \sum_{i=2}^{2^n-1} a_i = a_1 + \sum_{k=1}^{n-1} \sum_{i=2^k}^{2^{k+1}-1} a_i \leq a_1 + \sum_{k=1}^{n-1} 2^k a_{2^k} = a_1 + r_{n-1}.$$

and

$$r_n = \sum_{k=1}^n 2^k a_{2^k} \leq \sum_{k=1}^n 2 \sum_{i=2^{k-1}}^{2^k-1} a_i = 2 \sum_{i=1}^{2^n-1} a_i = 2s_{2^n-1}.$$

So, the (increasing) sequences r_n and $s_{2^{n-1}}$ converge or diverge simultaneously. But since (s_m) increases, it converges if its subsequence $s_{2^{n-1}}$ does. ■

Examples. Let $a > 0$.
(i) $\sum a^n n^\alpha$ converges if $a < 1$ and diverges if $a > 1$ for any $\alpha \in \mathbb{R}$ by the root test or the ratio test.
(ii) $\sum \frac{a^n n^\alpha}{n!}$ converges by the ratio test.
(iii) $\sum \frac{n! n^\alpha}{n^n}$ converges for any $\alpha \in \mathbb{R}$ by the ratio test.
(iv) $\sum \frac{1}{n^\alpha}$ converges if $\alpha > 1$ and diverges if $\alpha \leq 1$ by the integral test or the condensation test.
(v) $\sum \frac{1}{n(\log n)^\alpha}$ converges if $\alpha > 1$ and diverges if $\alpha \leq 1$ by the integral test or the condensation test.

The Riemann zeta function is defined by $\zeta(x) = \sum_{n=1}^\infty \frac{1}{n^x}$, which is defined for all $x > 1$. (Further, the zeta function can be *analytically continued* to the set $\mathbb{C}$ of complex numbers except 1, where it has *a pole*.)

8.3. Conditionally convergent series

If a series is convergent but is not absolutely convergent, it is said to be *conditionally convergent*.

There are tests for conditional convergence; the simplest one is the *Leibniz's alternating series test*:

Theorem 8.3.1. *If (a_i) is a sequence decreasing to 0 (that is, $a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0$ and $\lim a_i = 0$), then the series $\sum(-1)^{i-1} a_i$ converges.*

Proof. To make it clear, $\sum(-1)^{i-1} a_i = a_1 - a_2 + a_3 - a_4 + \cdots$ Put $s_n = \sum_{i=1}^n a_i$, $n \in \mathbb{N}$. I claim that $s_1 \geq s_3 \geq \cdots$, $s_2 \leq s_4 \leq \cdots$. Indeed, for any odd n, $s_{n+2} = s_n - a_{n+1} + a_{n+2} \leq s_n$ and for any even n, $s_{n+2} = s_n + a_{n+1} - a_{n+2} \geq s_n$. So, the sequence (s_{2k}) is increasing and either converges or diverges to $+\infty$; and the sequence (s_{2k+1}) is decreasing and either converges or diverges to $-\infty$; and also $s_n - s_{n-1} = a_n \longrightarrow 0$; so both (s_{2k}) and (s_{2k+1}) converge to the same limit, so the sequence (s_n) converges. ■

Examples. The series $\sum \frac{(-1)^i}{i}$, $\sum \frac{(-1)^i}{\sqrt{i}}$, and $\sum \frac{(-1)^i}{\log i}$ converge conditionally.

To get two more tests for conditional convergence, we will need *the summation by parts*, or *Abel's summation* formula:

Lemma 8.3.2. *Let $s_0, r_0, a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$, put $s_k = s_0 + \sum_{i=1}^k a_i$, $r_k = r_0 + \sum_{i=1}^k b_i$, $k = 1, \dots, n$. Then*

$$\sum_{i=1}^n s_i b_i = s_n r_n - s_1 r_0 - \sum_{i=1}^{n-1} a_{i+1} r_i.$$

Proof. For any i we have $b_i = r_i - r_{i-1}$, so

$$\begin{aligned}\sum_{i=1}^n s_i b_i &= \sum_{i=1}^n s_i(r_i - r_{i-1}) = \sum_{i=1}^n s_i r_i - \sum_{i=1}^n s_i r_{i-1} = \sum_{i=1}^n s_i r_i - \sum_{i=0}^{n-1} s_{i+1} r_i \\ &= \sum_{i=1}^{n-1} s_i r_i + s_n r_n - s_1 r_0 - \sum_{i=1}^{n-1} s_{i+1} r_i = s_n r_n - s_1 r_0 - \sum_{i=1}^{n-1} (s_{i+1} - s_i) r_i = s_n r_n - s_1 r_0 - \sum_{i=1}^{n-1} a_{i+1} r_i.\end{aligned}$$

■

An analog of Theorem 8.2.3 doesn't hold for conditionally converging series: the series $\sum (-1)^i/i$ converges, the sequence $((-1)^i)$ is bounded, but $\sum (-1)^i((-1)^i/i) = \sum (1/i)$ diverges.

We however have the following *Dirichlet's test*, generalizing Leibniz's test:

Theorem 8.3.3. *If $\sum b_i$ is a series with bounded partial sums and (a_i) is a sequence decreasing to 0, then the series $\sum a_i b_i$ converges.*

Proof. Put $a_0 = 0$ and $c_i = a_i - a_{i-1}$, $i \in \mathbb{N}$, so that $a_n = \sum_{i=1}^n c_i$, $n \in \mathbb{N}$. Since the sequence (a_n) is decreasing to 0, $c_i \leq 0$ for all $i \geq 2$, and $\sum_{i=2}^\infty |c_i| = a_1$. We are also given that the sequence $r_n = \sum_{i=1}^n b_i$, $n \in \mathbb{N}$, $r_0 = 0$, of partial sums of $\sum b_i$ is bounded. By the summation by parts formula, for any n,

$$\sum_{i=1}^n a_i b_i = a_n r_n - a_1 r_0 - \sum_{i=1}^{n-1} c_{i+1} r_i.$$

Since $a_n \longrightarrow 0$ and the sequence (r_n) is bounded, $a_n r_n \longrightarrow 0$. Since $\sum c_i$ converges absolutely and (r_i) is bounded, $\sum c_{i+1} r_i$ converges (absolutely). Hence, the sequence $\sum_{i=1}^n a_i b_i$ of partial sums of $\sum a_i b_i$ converges.

■

Example. Let $\alpha \in \mathbb{R}$, $\alpha \notin 2\pi\mathbb{Z}$; then for any $n \in \mathbb{N}$ we have $\left|\sin\alpha + \sin 2\alpha + \dots + \sin n\alpha\right| = \left|\frac{\cos(\alpha/2) - \cos((n+1/2)\alpha)}{2\sin(\alpha/2)}\right| \leq \frac{1}{|\sin(\alpha/2)|}$. So, the series $\sum \frac{\sin(n\alpha)}{n}$, $\sum \frac{\sin(n\alpha)}{\sqrt{n}}$, and $\sum \frac{\sin(n\alpha)}{\log n}$ converge conditionally.

And *Abel's test*:

Theorem 8.3.4. *If $\sum b_i$ is a convergent series and (a_i) is a bounded monotone sequence, then the series $\sum a_i b_i$ converges.*

Proof. W.l.o.g. let's assume that (a_i) is increasing, and let $a = \lim a_i$. Then $\sum a_i b_i = \sum (a_i - a) b_i + a \sum b_i$. Since $\sum b_i$ converges and $\sum (a_i - a) b_i$ converges by Dirichlet's test, $\sum a_i b_i$ converges. ■

Example. Since the series $\sum \frac{\sin(n\alpha)}{\log n}$ converges and the sequence $(1 + 1/n)^n$ is monotone and bounded, the series $\sum \frac{\sin(n\alpha)}{\log n}(1 + 1/n)^n$ also converges.

8.4. Groupings and rearrangements

Given a series $\sum_{i=1}^\infty a_i$ and a strictly increasing sequence $0 = k_0 < k_1 < k_2 < \cdots$ of integers, the series $\sum b_j$ where, for each j, $b_j = \sum_{i=k_{j-1}+1}^{k_j} a_i$, is called *a grouping* of $\sum a_i$.

If $\sum b_j$ is a grouping of $\sum a_i$, then the sequence of partial sums of $\sum b_j$ is a subsequence of the sequence of partial sums of $\sum a_i$, so, if $\sum a_i$ converges, then $\sum b_j$ also does, and to the same sum. The converse is not true: the series $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ diverges, whereas $(1-1) + (1-1) + (1-1) + \cdots$ converges. Here are two cases where grouping doesn't change the convergence:

Theorem 8.4.1. *If $a_i \geq 0$ for all i and $\sum b_j$ is a grouping of $\sum a_i$, then $\sum b_j < +\infty$ iff $\sum a_i < +\infty$.*

Proof. The sequence of partial sums of $\sum a_i$ is increasing, it converges iff it is bounded, which is true iff any of its subsequences is bounded. ■

Theorem 8.4.2. *Let $\sum a_i$ be a series such that $a_i \longrightarrow 0$ as $i \longrightarrow \infty$, let $0 = k_0 < k_1 < k_2 < \cdots$ be a strictly increasing sequence in $\mathbb{N}$ such that the sequence $(k_j - k_{j-1})$ is bounded, and let $b_j = \sum_{i=k_{j-1}+1}^{k_j} a_i$, $j \in \mathbb{N}$. Then the grouping $\sum b_j$ converges iff the series $\sum a_i$ converges.*

Proof. Let $s_n = \sum_{i=1}^n a_i$, $n \in \mathbb{N}$. If $\sum b_j$ converges, the subsequence (s_{k_m}) of (s_n) converges; let $s = \lim s_{k_m} (= \sum_{j=1}^\infty b_j)$. Let d be such that $k_j - k_{j-1} \le d$ for all j. Let $\varepsilon > 0$. Find l such that $|s_{k_m} - s| < \varepsilon/2$ for all $m \ge l$, find r such that $|a_i| < \varepsilon/(2d)$ for all $i \ge r$. Now let $n \ge \max\{k_l + 1, r + d\}$; find m such that $k_m + 1 \le n \le k_{m+1}$, then $m \ge l$, and $k_m \ge n - d \ge r$ so $|a_i| < \varepsilon/(2d)$ for all $i = k_m + 1, \ldots, n$, thus we get

$$\left|\sum_{i=1}^n a_i - s\right| = \left|\sum_{i=1}^{k_m} a_i - s\right| + \left|\sum_{i=k_m+1}^n a_i\right| \le |s_{k_m} - s| + \sum_{i=k_m+1}^n |a_i| < \varepsilon/2 + \varepsilon/(2d) \cdot d = \varepsilon.$$

So, $s_n \longrightarrow s$. ■

The next question is: Can the sum of a convergent series depend on the order of summation? The answer is *No* – if the series converges absolutely; *Yes* – if conditionally.

A rearrangement of a series $\sum_{i=1}^\infty a_i$ is the series $\sum\limits_{j=1}^{\infty} a_{i_j}$, where $j \mapsto i_j$ is a one-to-one correspondence $\mathbb{N} \longrightarrow \mathbb{N}$.

Example. $a_{101} + a_2 + a_{55} + a_1 + \cdots$ is a rearrangement of $a_1 + a_2 + a_3 + \cdots$ if for every i, the term a_i appears in the first formal sum exactly once.

And, the sum of the rearrangement of a series may differ from the sum of the original series!

Example. Let $s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ (actually, $S = \log 2$). Consider the following rearrangement of this series: $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots$; after some "grouping" (which doesn't affect the sum), it is

$$\left(1 - \tfrac{1}{2}\right) - \tfrac{1}{4} + \left(\tfrac{1}{3} - \tfrac{1}{6}\right) - \tfrac{1}{8} + \cdots = \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} - \tfrac{1}{8} + \cdots = \tfrac{1}{2}\left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots\right) = \tfrac{1}{2}s.$$

Theorem 8.4.3. *If a series $\sum a_i$ is absolutely convergent, then any its rearrangement $\sum a_{i_j}$ also converges and $\sum_{j=1}^\infty a_{i_j} = \sum_{i=1}^\infty a_i$.*

Proof. Let $s = \sum_{i=1}^\infty a_i$. Let $\varepsilon > 0$. Find k such that $\left|\sum_{i=1}^k a_i - s\right| < \varepsilon/2$ and (using the Cauchy criterion for $\sum |a_i|$) for any $n > m \ge k$, $\sum_{i=m+1}^n |a_i| < \varepsilon/2$. Given a rearrangement $\sum a_{i_j}$ of $\sum a_i$, find m such that $1, 2, \ldots, k \in \{i_1, \ldots, i_m\}$. Let $l \ge m$, let $d = \max\{i_j,\ j \le l\}$, then

$$\left|\sum_{j=1}^l a_{i_j} - s\right| = \left|\sum_{i=1}^k a_i + \sum_{\substack{1 \le j \le l \\ i_j > k}} a_{i_j} - s\right| \le \left|\sum_{i=1}^k a_i - s\right| + \sum_{\substack{1 \le j \le l \\ i_j > k}} |a_{i_j}| \le \left|\sum_{i=1}^k a_i - s\right| + \sum_{i=k+1}^d |a_i| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So, $\sum_{j=1}^\infty a_{i_j} = s$. ■

Theorem 8.4.4. *Let $\sum a_i$ be a series, let (b_i) be the subsequence of the nonnegative terms of (a_i), and let (c_i) be the subsequence of the negative terms of (a_i).*
(i) *If both $\sum b_i$ and $\sum c_i$ converge, then $\sum a_i$ is absolutely convergent and $\sum_{i=1}^\infty a_i = \sum_{i=1}^\infty b_i + \sum_{i=1}^\infty c_i$.*
(ii) *If $\sum b_i$ diverges and $\sum c_i$ converges, then $\sum a_i$ diverges to $+\infty$.*
(iii) *If $\sum b_i$ converges and $\sum c_i$ diverges, then $\sum a_i$ diverges to $-\infty$.*
(iv) *If both $\sum b_i$ and $\sum c_i$ diverge, then $\sum a_i$ is either conditionally convergent or divergent.*

Proof. (i) Let $s = \sum b_i < +\infty$ and $r = \sum c_i > -\infty$. Let $\varepsilon > 0$; find k such that for any $n \ge k$, $\left|\sum_{i=1}^n b_i - s\right| < \varepsilon/2$ and $\left|\sum_{i=1}^n c_i - r\right| < \varepsilon/2$. Find l such that the (finite) sequence $(a_1, \ldots, a_l)$ contains all $b_1, \ldots, b_k, c_1, \ldots, c_k$. Then for any $n \ge l$, $\sum_{i=1}^n |a_i| = \sum_{i=1}^{n_1} b_i + \sum_{i=1}^{n_2} (-c_i)$ with $n_1, n_2 \ge k$, so $\left|\sum_{i=1}^n |a_i| - (s - r)\right| < \varepsilon$; hence, $\sum_{i=1}^\infty |a_i| = s - r$. Also, $\sum_{i=1}^n a_i = \sum_{i=1}^{n_1} b_i + \sum_{i=1}^{n_2} c_i$ so $\left|\sum_{i=1}^n a_i - (s + r)\right| < \varepsilon$; hence, $\sum_{n=1}^\infty a_i = s + r$.
(ii) Let $\sum b_i = +\infty$ diverge and $r = \sum c_i > -\infty$. Let $N \in \mathbb{R}$; find k such that $\sum_{i=1}^n b_i > N - r$. Find l such that the sequence $(a_1, \ldots, a_l)$ contains all $b_1, \ldots, b_k$. Then for any $n \ge l$, $\sum_{i=1}^n a_i = \sum_{i=1}^{n_1} b_i + \sum_{i=1}^{n_2} c_i$ for some $n_1 \ge k$ and some n_2, so $\sum_{i=1}^n a_i > N - r + r = N$. Thus $\sum a_i$ diverges to $+\infty$.

(iii) is similar to (ii).
(iv) If both $\sum b_i = +\infty$ and $\sum c_i = -\infty$, then $\sum a_i$ cannot converge absolutely, so either it converges conditionally or diverges. $\blacksquare$

The following nice (but useless) fact is known as *the Riemann rearrangement theorem*:

Theorem 8.4.5. *If a series $\sum a_i$ converges conditionally then for any $s \in \mathbb{R} \cup \{-\infty, +\infty\}$ there exists a rearrangement $\sum a_{i_j}$ of $\sum a_i$ with $\sum_{j=1}^{\infty} a_{i_j} = s$.*

Proof. let (b_i) be the subsequence of the nonnegative terms of (a_i), and let (c_i) be the subsequence of the negative terms of (a_i); since $\sum a_i$ converges, $\sum b_i = +\infty$, $\sum c_j = -\infty$, and $b_i, c_i \longrightarrow 0$.

Let $s \in \mathbb{R}$. Let's rearrange $\sum a_i$ in the following way: find the minimal k_1 such that $r_1 = b_1 + \cdots + b_{k_1} > s$, then $0 < r_1 - s \le b_{k_1}$; then find the minimal l_1 such that $t_1 = r_1 + c_1 + \cdots + c_{l_1} < s$, then $0 < s - l_1 \le -c_{l_1}$; find the minimal k_2 such that $r_2 = t_1 + b_{k_1+1} + \cdots + b_{k_2} > s$, then $0 < r_2 - s \le b_{k_2}$; then find the minimal l_2 such that $t_2 = r_2 + c_{l_1+1} + \cdots + c_{l_2} < s$, then $0 < s - l_2 \le -c_{l_2}$; and so on. Since $b_{k_j}, c_{l_j} \longrightarrow 0$, both sequences (r_j) and (t_j) converge to s. The partial sums of the rearrangement

$$b_1 + \cdots + b_{k_1} + c_1 + \cdots + c_{l_1} + b_{k_1+1} + \cdots + b_{k_2} + c_{l_1+1} + \cdots + c_{l_2} + \cdots.$$

of $\sum a_i$ oscillate between $r_1, s_1, r_2, s_2, \ldots$, thus also converge to s.

For $s = +\infty$, we rearrange $\sum a_i$ in the following way: for every j choose k_j such that $b_1 + \cdots + b_{k_j} > j - c_1 - \cdots - c_j$, then for any $n \ge k_j$, the n-th partial sum of the series

$$b_1 + \cdots + b_{k_1} + c_1 + b_{k_1+1} + \cdots + b_{k_2} + c_2 + \cdots$$

is $\ge j$, so the series diverges to $+\infty$.

The case $s = -\infty$ is similar. $\blacksquare$

8.5. Double series and unordered sums

A double series is a formal sum $\sum a_{i,j}$, $a_{i,j} \in \mathbb{R}$. The sum of such a series may depend on the *order of summation*: the sum can be defined as $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j}$, or as $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{i,j}$, or as $\sum_{k=1}^{\infty} \sum_{i=1}^{k} a_{i,k+1-i}$, or as $\sum_{k=1}^{\infty} \sum_{i,j:\max(i,j)=k} a_{i,j}$, etc. However, we will show that if the double series converges absolutely, that is, if $\sum |a_{i,j}| < +\infty$ with respect to at least one of the order of summation, then the series $\sum a_{i,j}$ converges to the same sum with respect to any other of summation.

Let's consider a more general situation. Let Λ be a "set of indices" (that is, just any set), and $i \mapsto a_i$ be a mapping from Λ to $\mathbb{R}$. What is $\sum_{i \in \Lambda} a_i$?

First, suppose that $a_i \ge 0$ for all i. We don't know what "an order of summation" is, but with respect to any such order we must have $\sum_{i \in P_1} a_i \le \sum_{i \in P_2} a_i$ whenever $P_1 \subseteq P_2 \subseteq \Lambda$. Thus under any order of summation, for any finite $F \subseteq \Lambda$ we must have $\sum_{i \in F} a_i \le \sum_{i \in \Lambda} a_i$. So, we just define $\sum_{i \in \Lambda} a_i$ as the supremum of the set of all finite sums of a_i, $\sum_{i \in \Lambda} a_i = \sup\bigl\{\sum_{i \in F} a_i,\ F \subseteq \Lambda,\ |F| < +\infty\bigr\}$. This supremum may be infinite; if $s = \sum_{i \in \Lambda} a_i < +\infty$, then $\sum_{i \in \Lambda} a_i$ is said to be *converging*, or *summable*. If $s = \sum_{i \in \Lambda} a_i < +\infty$, then for any $\varepsilon > 0$ there exists a finite set $F \subseteq \Lambda$ such that $\sum_{i \in F} a_i > s - \varepsilon$, and then $\sum_{i \in \Lambda \setminus F} a_i < \varepsilon$.

In the general case, where a_i are not assumed to be nonnegative, it is not clear how to define $\sum_{i \in \Lambda} a_i$. I claim however that if this sum "converges absolutely", that is, if $\sum_{i \in \Lambda} |a_i| < \infty$, all "reasonable" definitions of $\sum_{i \in \Lambda} a_i$ give the same result. For every $P \subseteq \Lambda$ let $\mathbf{S}_{i \in P} a_i \subseteq \mathbb{R} \cup \{+\infty\}$ be *the set* of "all limit points of the set of partial sums for a collection of methods of summation" of a_i, $i \in P$. This is not a good rigorous definition, let's define these sets "axiomatically": assume that a collection of sets $\mathbf{S}_{i \in P} a_i \subseteq \mathbb{R} \cup \{+\infty\}$, $P \subseteq \Lambda$, satisfies the following conditions:
(σ_1) $\mathbf{S}_{i \in \emptyset} a_i = 0$.
(σ_2) For any $P \subset \Lambda$ and any $j \in \Lambda \setminus P$, $\mathbf{S}_{i \in P \cup \{j\}} a_i = a_j + \mathbf{S}_{i \in P} a_i$ $(= \bigl\{a_j + r,\ r \in \mathbf{S}_{i \in P} a_i\bigr\})$.
(σ_3) For any $P \subseteq \Lambda$ for every $r \in \mathbf{S}_{i \in P} a_i$, $|r| \le \sum_{i \in P} |a_i|$.
It follows by induction from (σ_1) and (σ_2) that for any finite set $F \subseteq \Lambda$, $\mathbf{S}_{i \in F} a_i = \bigl\{\sum_{i \in F} a_i\bigr\}$, and for any $P \subseteq \Lambda$ with $P \cap F = \emptyset$, $\mathbf{S}_{i \in P \cup F} a_i = \sum_{i \in F} a_i + \mathbf{S}_{i \in P} a_i$.

Examples. (i) Let $(a_i)_{i \in \mathbb{N}}$ be a sequence; for every $P \subseteq \mathbb{N}$ let $\mathbf{S}_{i \in P} a_i$ be the set of all the limit points of the set $\bigl\{\sum_{i \in F} a_i : F \subseteq P,\ |F| < \infty\bigr\}$. It is easy to check that the sets $\mathbf{S}_{i \in P} a_i$ satisfy (σ_1)-(σ_3).

(ii) Let $(a_{i,j})_{(i,j)\in\mathbb{N}^2}$ be a double series. Let $P \in \mathbb{N}^2$; for every i let $P_i = \{j : (i,j) \in P\}$. For every $i \in \mathbb{N}$ let S_i be the set of all the limit points of the set $\big\{\sum_{j\in F_i} a_{i,j} : F_i \subseteq P_i,\ |F_i| < \infty\big\}$. We then define $\mathbf{S}_{(i,j)\in P}\, a_i$ as the set of all the limit points of $\big\{\sum_{i\in F} s_i : F \subseteq \mathbb{N},\ |F| < \infty,\ s_i \in S_i,\ i \in F\big\}$.

Under the (very mild) assumptions (σ_1)-(σ_3), we can prove that if $\sum_{i\in\Lambda} a_i$ *converges absolutely*, that is, $\sum_{i\in\Lambda} |a_i| < +\infty$, then $\mathbf{S}_{i\in\Lambda}\, a_i$ consists of a single real number:

Theorem 8.5.1. *If $\sum_{i\in\Lambda} |a_i| < +\infty$, then $\mathbf{S}_{i\in\Lambda}\, a_i$ is a sungleton, $\mathbf{S}_{i\in\Lambda}\, a_i = \{s\}$ for some $s \in \mathbb{R}$.*

Proof. For each $n \in \mathbb{N}$ choose a finite set $F_n \subseteq \Lambda$ such that for $P_n = \Lambda \setminus F_n$ we have $\sum_{i\in P_n} |a_i| < 1/n$, and let $s_n = \sum_{i\in F_n} a_i$. Then for any $n, m \in \mathbb{N}$, since $F_n \setminus F_m \subseteq P_m$ and $F_m \setminus F_n \subseteq P_n$, we have

$$|s_n - s_m| = \Big|\sum_{i\in F_n} a_i - \sum_{i\in F_m} a_i\Big| \le \Big|\sum_{i\in F_n\setminus F_m} a_i\Big| + \Big|\sum_{i\in F_m\setminus F_n} a_i\Big| \le \sum_{i\in F_n\setminus F_m} |a_i| + \sum_{i\in F_m\setminus F_n} |a_i| < 1/m + 1/n.$$

This shows that the sequence (s_n) is Cauchy; thus, it converges to some $s \in \mathbb{R}$.

Now let $r \in \mathbf{S}_{i\in\Lambda}\, a_i$. Given $\varepsilon > 0$, find n such that $|s_n - s| < \varepsilon/2$ and $1/n < \varepsilon/2$; then by (σ_2) we have that $r = s_n + t$ for some $t \in \mathbf{S}_{i\in P_n}\, a_i$ and so by (σ_3), $|t| < \varepsilon/2$. Hence, $|r - s| < \varepsilon$. Since ε was arbitrary, $r = s$. ■

If $\sum_{i\in\Lambda} |a_i| < +\infty$, we denote the (unique) number s such that $\mathbf{S}_{i\in\Lambda}\, a_i = \{s\}$ by $\sum_{i\in\Lambda} a_i$ and call it *the sum* of a_i, $i \in \Lambda$. Note that if $a_i \ge 0$ for all $i \in \Lambda$, then under this new definition, $\sum_{i\in\Lambda} a_i = \sup\big\{\sum_{i\in F} a_i,\ F \subseteq \Lambda,\ |F| < +\infty\big\}$ as before.

Let's summarize: let $i \mapsto a_i$ be a mapping $\Lambda \longrightarrow \mathbb{R}$. Then

(i) If $a_i \ge 0$ for all i then "all orders of summation" of the numbers a_i, $i \in \Lambda$, give the same result, namely, $\sup\big\{\sum_{i\in F} a_i,\ F \subseteq \Lambda,\ |F| < \infty\big\}$, finite or infinite which we denote by $\sum_{i\in\Lambda} a_i$.

(ii) If $\sum_{i\in\Lambda} |a_i| < \infty$ then "all orders of summation" of the numbers a_i, $i \in \Lambda$, give the same finite result, which we denote by $\sum_{i\in\Lambda} a_i$. We say that $\sum a_i$ converges absolutely in this case.

(iii) If $\sum a_i$ doesn't converge absolutely, $\sum_{i\in\Lambda} |a_i| = \infty$, then $\sum_{i\in\Lambda} a_i$ is not well defined: like in the Riemann rearrangement theorem, it depends on the "order of summation" and, under some such orders, may not exit at all.

To determine if a sum $\sum a_i$ converges absolutely we check if $\sum_{i\in\Lambda} |a_i| < \infty$ with respect to an arbitrary order of summation.

As an example, let's obtain the following result:

Theorem 8.5.2. *If series $\sum a_i$ and $\sum b_j$ are absolutely convergent then the double series $\sum a_i b_j$ is also absolutely convergent, and $\sum_{k=2}^{\infty}\sum_{i=1}^{k-1} a_i b_{k-i} = \sum_{i=1}^{\infty} a_i \cdot \sum_{i=1}^{\infty} b_i$.*

Proof. If $\sum_{i=1}^{\infty} |a_i|, \sum_{i=1}^{\infty} |b_i| < \infty$, then $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_i b_j| = \sum_{i=1}^{\infty} |a_i| \sum_{j=1}^{\infty} |b_j| < \infty$, so, the double series $\sum a_i b_j$ converges absolutely. Hence, the sum of the series doesn't depend on the method of summation, and so, $\sum_{k=2}^{\infty}\sum_{i=1}^{k-1} a_i b_{k-i} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i b_j = \sum_{i=1}^{\infty} a_i \cdot \sum_{j=1}^{\infty} b_j$. ■

The sum $\sum_{k=1}^{\infty}\sum_{i=1}^{k-1} a_i b_{k-i}$ is called *the Cauchy product* of the series $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$.

8.6. Infinite products

Let $a_i > 0$ for all i; we say that the infinite product $\prod a_i$ *converges* if the sequence $p_n = \prod_{i=1}^{n} a_i$, $n \in \mathbb{N}$, of its *partial products* converges to $p \neq 0$, in which case we write $\prod_{i=1}^{\infty} a_i = p$.

Examples. (i) The product $\prod_{i=1}^{\infty} \frac{i+1}{i}$ diverges to ∞, $\prod_{i=1}^{\infty} \frac{i}{i+1}$ diverges to 0, $\prod_{i=2}^{\infty} \frac{i^2-1}{i^2} = \frac{1}{2}$.

(ii) For any $\alpha \in \mathbb{R}$, $\sin\alpha = \alpha \prod_{i=1}^{\infty}\big(1 - \frac{\alpha^2}{i^2\pi^2}\big)$. (The proof requires complex analysis.)

(iii) The product $\prod_{i=1}^{\infty}\big(1 - \frac{\alpha}{i\pi}\big)$ diverges to 0.

(iv) Let $p_1, p_2, \ldots \in \mathbb{N}$ be the sequence of all prime integers; then for every $\alpha > 0$, $\prod_{i=1}^{\infty} \frac{1}{1-p_i^{-\alpha}} = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$. Indeed, for every $k \in \mathbb{N}$ let P_k be the set of natural numbers having no prime divisors other than $p_1, \ldots, p_k$, that is, of the form $p_1^{r_1} \cdots p_k^{r_k}$, $r_1, \ldots, r_k \geq 0$. Since (by the Fundamental Theorem of Arithmetic) for every $n \in P_k$ such a representation is unique,

$$\prod_{i=1}^{k} \frac{1}{1-p_i^{-\alpha}} = \prod_{i=1}^{k} \left(1 + p_i^{-\alpha} + p_i^{-2\alpha} + \cdots\right) = \prod_{i=1}^{k} \lim_{d\to\infty} \sum_{r=0}^{d} p_i^{-r\alpha} = \lim_{d\to\infty} \prod_{i=1}^{k} \sum_{r=0}^{d} p_i^{-r\alpha}$$
$$= \lim_{d\to\infty} \sum_{0\leq r_1,\ldots,r_k\leq d} \prod_{i=1}^{k} p_i^{-r_i\alpha} = \lim_{d\to\infty} \sum_{0\leq r_1,\ldots,r_k\leq d} \Big(\prod_{i=1}^{k} p_i^{-r_i}\Big)^{\alpha} = \sum_{n\in P_k} n^{-\alpha}.$$

(I used the fact that the limit of the product of finitely many sequences is the product of the limits of these sequences. And since the numbers $n^{-\alpha}$ are positive, the order of their summation doesn't matter.) Since, by the Fundamental Theorem of Arithmetic, $\bigcup_{k=1}^{\infty} P_k = \mathbb{N}$, we obtain

$$\prod_{i=1}^{\infty} \frac{1}{1-p_i^{-\alpha}} = \lim_{k\to\infty} \prod_{i=1}^{k} \frac{1}{1-p_i^{-\alpha}} = \lim_{k\to\infty} \sum_{n\in P_k} n^{-\alpha} = \sum_{n\in\mathbb{N}} n^{-\alpha} = \sum_{n=1}^{\infty} n^{-\alpha}.$$

Hence, $\prod_{i=1}^{\infty} \frac{1}{1-p_i^{-\alpha}} = \zeta(\alpha) \in \mathbb{R}$ for $\alpha > 1$ (where ζ is the zeta function) and $= +\infty$ for $0 < \alpha \leq 1$.

Theorem 8.6.1. *If a product $\prod a_i$ converges, then $a_i \longrightarrow 1$.*

Proof. If $\prod_{i=1}^{n} a_i \longrightarrow p \neq 0$, then $a_n = \prod_{i=1}^{n} a_i / \prod_{i=1}^{n-1} a_i \longrightarrow p/p = 1$. ■

Theorem 8.6.2. *A product $\prod a_i$ converges iff the series $\sum \log a_i$ converges, and then $\prod_{i=1}^{\infty} a_i = e^{\Sigma_{i=1}^{\infty} \log a_i}$.*

Proof. Let $p_n = \prod_{i=1}^{n} a_i$, $n \in \mathbb{N}$; then $s_n = \log p_n = \sum_{i=1}^{n} \log a_i$. Since log is a continuous function, if $p_n \longrightarrow p > 0$ then $s_n = \log p_n \longrightarrow \log p$. Since exp is a continuous function, if $s_n \longrightarrow s$ then $p_n = e^{s_n} \longrightarrow e^s$. ■

Theorem 8.6.3. *If $x_i \geq 0$ for all i, then the product $\prod(1 + x_i)$ converges iff the series $\sum x_i$ converges.*

Proof. By Theorem 8.6.2, $\prod(1 + x_i)$ converges iff the series $\sum \log(1 + x_i)$ converges. Now, if $\prod(1 + x_i)$ converges or $\sum x_i$ converges, then $x_i \longrightarrow 0$, and so, $\log(1 + x_i)/x_i \longrightarrow 1$ as $i \longrightarrow \infty$. Hence, since $x_i \geq 0$ and $\log(1 + x_i) > 0$ for all i, by the limit comparison test $\sum \log(1 + x_i)$ converges iff $\sum x_i$ converges. ■

Example. Let (p_i) be the sequence of the prime integers. Since $\prod_{i=1}^{\infty} \frac{1}{1-p_i^{-1}} = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$, we also have

$$+\infty = \sum_{i=1}^{\infty} \Big(\frac{1}{1-p_i^{-1}} - 1\Big) = \sum_{i=1}^{\infty} \Big(\frac{1}{p_i} \cdot \frac{1}{1-p_i^{-1}}\Big).$$

Since $\frac{1}{1-p_i^{-1}} \longrightarrow 1$ we obtain by the limit comparison test that $\sum_{i=1}^{\infty} \frac{1}{p_i} = +\infty$.

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9. Functional sequences and series

9.1. Uniform convergence of functional sequences

A sequence of functions (f_n) from a set A to a set B is a mapping from $\mathbb{N}$ into the set $\{f\colon A \longrightarrow B\}$, $n \mapsto f_n$. We say that a sequence (f_n) of functions $A \longrightarrow \mathbb{R}$ *converges* or *converges pointwise* to a function f, and write $f_n \longrightarrow f$, if $\lim f_n(x) = f(x)$ for all $x \in A$.

Examples. (i) The sequence $f_n = 1/n$, $n \in \mathbb{N}$, converges to 0 on $\mathbb{R}$.
(ii) Let $f_n(x) = x/n$, $x \in \mathbb{R}$, $n \in \mathbb{N}$; then $f_n \longrightarrow 0$ on $\mathbb{R}$.
(iii) Let $f_n(x) = x^n$, $x \in [0, 1]$, $n \in \mathbb{N}$; then $f_n \longrightarrow f$ where $f(x) = \begin{cases} 0, & x \in [0, 1) \\ 1, & x = 1 \end{cases}$.

(iv) Let $f_n(x) = \begin{cases} 0, & x = 0 \\ n, & 0 < x < 1/n \\ 0, & 1/ \leq x \leq 1 \end{cases}$. Then $f_n \longrightarrow 0$.

(v) Let $f_n(x) = \begin{cases} 0, & 0 < x < 1/n \\ 1/x, & 1/n \leq x \end{cases}$, $n \in \mathbb{N}$. Then $f_n \longrightarrow f(x) = 1/x$ on $(0, +\infty)$.

(vi) $P_{0,n,\exp} \longrightarrow \exp$ and $P_{0,n,\sin} \longrightarrow \sin$ on $\mathbb{R}$, $P_{0,n,\log} \longrightarrow \log$ on $(-1, 1]$.

Notice that in (iii) a sequence of continuous functions converges to a discontinuous function; in (iv), $f_n \longrightarrow 0$ but $\int_0^1 f_n = 1 \not\longrightarrow 0 = \int_0^1 0$; in (v), a sequence of bounded functions converges to an unbounded function. This shows that pointwise convergence "is weak": it doesn't guarantee that the limit function inherits good properties from the members of the sequence.

We say that (f_n) *converges to f uniformly on A* and write $f_n \rightrightarrows f$ if for any $\varepsilon > 0$ there exists k such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq k$ for all $x \in A$. Clearly, uniform convergence implies pointwise convergence.

Examples. (i) The sequence of constant functions $f_n(x) = 1/n$ converges to 0 uniformly on $\mathbb{R}$.

(ii) The sequence $f_n(x) = \frac{1}{n}\sin(nx)$ also converges to 0 uniformly on $\mathbb{R}$.

(iii) The sequence $f_n(x) = e^{x/n}$ converges to 0 not uniformly on $\mathbb{R}$ but uniformly on the interval $(-\infty, b]$ for any $b \in \mathbb{R}$.

The uniform norm $\|f\|_A$, or simply $\|f\|$, of a function $f\colon A \longrightarrow \mathbb{R}$ is $\sup\big\{|f(x)| : x \in A\big\}$. $\|f\|_A$ may be infinite; $\|f\|_A < +\infty$ iff f is bounded on A.

$\|f - g\|_A$ plays the role of a "distance" between functions f and g:

Theorem 9.1.1. *For any functions $f, g, h\colon A \longrightarrow \mathbb{R}$ one has:*

(i) $\|f\| \geq 0$, *and* $\|f\| = 0$ *iff* $f = 0$ *on* A;

(ii) $\|f + g\| \leq \|f\| + \|g\|$;

(iii) $\|fg\| \leq \|f\| \cdot \|g\|$;

(iv) $\|cf\| = |c| \cdot \|f\|$ *for any* $c \in \mathbb{R}$;

(v) $\|f - g\| \leq \|f - h\| + \|g - h\|$.

Proof. (i) is evident.

(ii) For any $x \in A$, $|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\|$, so $\|f + g\| = \sup\{|f(x) + g(x)|,\ x \in A\} \leq \|f\| + \|g\|$.

(iii) For any $x \in A$, $|f(x)g(x)| = |f(x)| \cdot |g(x)| \leq \|f\| \cdot \|g\|$, so $\|fg\| = \sup\{|f(x)g(x)|,\ x \in A\} \leq \|f\| \cdot \|g\|$.

(iv) For any $x \in A$, $|cf(x)| = |c| \cdot |f(x)|$, so $\|cf\| = \sup\{|cf(x)|,\ x \in A\} = |c|\sup\{|f(x)|,\ x \in A\} = |c| \cdot \|f\|$.

(v) follows from (ii). ■

Now, the uniform convergence of functional sequences is just convergence with respect to uniform norm:

Theorem 9.1.2. $f_n \rightrightarrows f$ *on* A *iff* $\lim \|f_n - f\|_A = 0$.

Example. To prove that the sequence $f_n(x) = x^{1+1/n}$ converges to $f(x) = x$ uniformly on $[0, 1]$ we need to find (or estimate) the norm $\|f_n - f\|$. For every n, the function $f_n - f$ is ≤ 0, $= 0$ at 0 and 1, and so, reaches it minimal value at the point x_n where $(f_n - f)'(x_n) = 0$, namely at $x_n = \frac{1}{(1+1/n)^n}$. So,

$$\|f_n - f\| = \Big|\Big(\frac{1}{(1+1/n)^n}\Big)^{1+1/n} - \frac{1}{(1+1/n)^n}\Big| = \frac{1}{(1+1/n)^n} \cdot \Big|\frac{1}{1+1/n} - 1\Big| \longrightarrow e^{-1} \cdot 0 = 0.$$

Another way to prove that $f_n \rightrightarrows f$ is to use *Dini's theorem*:

Theorem 9.1.3. *Suppose that (f_n) is a monotone sequence of continuous functions on a closed bounded interval I that converges pointwise to a continuous function f. Then $f_n \rightrightarrows f$.*

Proof. W.l.o.g. we may assume that the sequence (f_n) decreases to f, $f_1(x) \geq f_2(x) \geq \cdots \geq f(x)$ and $f_n(x) \longrightarrow f(x)$ for all $x \in I$. Assume that f_n do not converge to f uniformly, then there is $\varepsilon > 0$ and a subsequence (f_{n_k}) of (f_n) such that for every k there is a point $x_k \in I$ such that $f_{n_k}(x_k) > f(x_k) + \varepsilon$. By Bolzano-Weierstrass's theorem, there exists a subsequence (x_{k_i}) of (x_k) that converges to a point $x_0 \in I$. Now, for every n, $f_n(x_{k_i}) \geq f_{n_{k_i}}(x_{k_i}) \geq f(x_{k_i}) + \varepsilon$ for all i such that $n_{k_i} \geq n$, so, by continuity of f_n and f, $f_n(x_0) = \lim_{i\to\infty} f_n(x_{k_i}) \geq \lim_{i\to\infty} f(x_{k_i}) + \varepsilon = f(x_0) + \varepsilon$. Hence, $f_n(x_0) \not\longrightarrow f(x_0)$, contradiction. ■

The Cauchy criterion for uniform convergence is:

Theorem 9.1.4. *Let (f_n) be a sequence of functions on a set A. Then (f_n) converges uniformly on A iff $\|f_n - f_m\|_A \longrightarrow 0$ as $n, m \longrightarrow \infty$ (that is, for any $\varepsilon > 0$ there exists k such that $\|f_n - f_m\| < \varepsilon$ for all $n, m \geq k$).*

Proof. If $f_n \rightrightarrows f$, then $\|f_n - f\| \longrightarrow 0$, so $\|f_n - f_m\| \leq \|f_n - f\| + \|f_m - f\| \longrightarrow 0$ as $n, m \longrightarrow \infty$.

Conversely, assume that $\|f_n - f_m\| \longrightarrow 0$ as $n, m \longrightarrow \infty$. For any $x \in A$, $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|$, so the sequence $(f_n(x))$ is Cauchy, so converges; define $f(x) = \lim_{n\to\infty} f_n(x)$, then $f_n \longrightarrow f$ pointwise.

Let $\varepsilon > 0$; find k such that $\|f_n - f_m\| < \varepsilon$ for all $n, m \geq k$, then for any $x \in A$, $|f_n(x) - f_m(x)| < \varepsilon$. Fix $n \geq k$, then $|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \varepsilon$ for all $x \in A$; so $\|f_n - f\| \leq \varepsilon$. Hence, $f_n \rightrightarrows f$. ■

Uniform convergence "preserves many good properties" of the functions of a sequence. We say that a sequence (f_n) of functions on a set A is *uniformly bounded on A* if there exists $M \in \mathbb{R}$ such that $|f_n(x)| \leq M$ for all $x \in A$ and all n, that is, $\|f_n\| \leq M$ for all n.

Theorem 9.1.5. *Let $f_n \rightrightarrows f$ on a set A. If f_n is bounded on A for some n large enough (that is, there is k such that if f_n is bounded for some $n \geq k$), then f is also bounded on A. If f is bounded on A, then for some k, the sequence $f_k, f_{k+1}, \ldots$ is uniformly bounded on A.*

Proof. Find k such that $\|f_n - f\| \leq 1$ for all $n \geq k$. If f_n is bounded for such an n, then $\|f\| \leq \|f_n\| + 1 < +\infty$, so f is bounded. If f is bounded, then $\|f_n\| \leq \|f\| + 1$ for all $n \geq k$, so the sequence $(f_k, f_{k+1}, \ldots)$ is uniformly bounded. ■

Theorem 9.1.6. *If $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$ on a set A, then $f_n + g_n \rightrightarrows f + g$ on A. If, in addition, f and g are bounded on A, then also $f_n g_n \rightrightarrows fg$ on A.*

Proof. $\big\|(f_n + g_n) - (f + g)\big\| \leq \|f_n - f\| + \|g_n - g\| \longrightarrow 0$.

If f and g are bounded on A then the sequences (f_n) and (g_n) are (eventually) uniformly bounded, so the sequences $(\|f_n\|)$ and $(\|g_n\|)$ are (eventually) bounded, so $\|f_n g_n - fg\| \leq \|f_n g_n - f g_n\| + \|f g_n - fg\| \leq \|f_n - f\| \cdot \|g_n\| + \|f\| \cdot \|g_n - g\| \longrightarrow 0$. ■

Theorem 9.1.7. *If $f_n \rightrightarrows f$ on a set A, $a \in A$, and all f_n are continuous at a, then $f_{|A}$ is continuous at a.*

Proof. Let $\varepsilon > 0$. Find n such that $\|f_n - f\| < \varepsilon/3$, so $|f_n(x) - f(x)| < \varepsilon/3$ for all $x \in A$. Find $\delta > 0$ such that $|f_n(x) - f_n(a)| < \varepsilon/3$ whenever $x \in A$, $|x - a| < \delta$. Then for any such x,

$$|f(x) - f(a)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

So, f is continuous at a. ■

Corollary 9.1.8. *If $f_n \rightrightarrows f$ on A and all f_n are continuous on A, then $f_{|A}$ is continuous.*

Theorem 9.1.9. *If $f_n \rightrightarrows f$ on an interval $[a, b]$ and all f_n are integrable on $[a, b]$, then f is also integrable on $[a, b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.*

Proof. Let $\varepsilon > 0$. Find n such that $\|f_n - f\| < \varepsilon/(3(b-a))$. Find any partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ such that $\Delta(f_n, P) < \varepsilon/3$. We then have $\big|U(f,P) - U(f_n,P)\big| < \sum_{i=1}^n \frac{\varepsilon}{3(b-a)}\Delta x_i = \frac{\varepsilon}{3(b-a)}(b-a) = \varepsilon/3$ and $\big|L(f,P) - L(f_n,P)\big| < \varepsilon/3$, so $\Delta(f,P) = U(f,P) - L(f,P) \le U(f_n,P) + \varepsilon/3 - L(f_n,P) + \varepsilon/3 = \Delta(f_n,P) + 2\varepsilon/3 = \varepsilon$. Hence, f is integrable.

Let $\varepsilon > 0$. Find k such that $\|f_n - f\| < \varepsilon/(b-a)$ for all $n \ge k$. Then for any $n \ge k$,

$$\Big|\int_a^b f_n - \int_a^b f\Big| \le \int_a^b |f_n - f| \le \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

So, $\int_a^b f_n \longrightarrow \int_a^b f$. ■

Theorem 9.1.10. *If $f_n \rightrightarrows f$ on a bounded interval I, f_n are integrable for all n, $x_0 \in I$, and $F_n(x) = \int_{x_0}^x f_n$, $x \in I$, then $F_n \rightrightarrows F$ on I where $F(x) = \int_{x_0}^x f$, $x \in I$. If, additionally, f_n for all n and f have primitives G_n and G respectively (this is so, for example, if f_n are continuous) and $G_n(x_0) \longrightarrow G(x_0)$, then $G_n \rightrightarrows G$.*

Proof. Since f_n are integrable, f is also integrable. Let $\varepsilon > 0$; find k such that $\|f_n - f\| < \varepsilon/|I|$ for all $n \ge k$. Then for any $n \ge k$, for any $x \in I$,

$$\big|F_n(x) - F(x)\big| = \Big|\int_{x_0}^x f_n - \int_{x_0}^x f\Big| \le \int_{x_0}^x |f_n - f| \le \frac{\varepsilon}{|I|}|x - x_0| < \varepsilon.$$

So, $F_n \rightrightarrows F$.

If G_n are G are such that $G_n' = f_n$, $n \in \mathbb{N}$, and $G' = f$, then by the F.T.C., for every n, $G_n = F_n + G_n(x_0)$. So, $G_n \rightrightarrows F + G(x_0) = G$. ■

Uniform convergence doesn't support differentiation: the sequence $f_n(x) = \frac{1}{n}\sin(nx)$ converges to 0 uniformly on $\mathbb{R}$, but the sequence $f_n'(x) = \cos(nx)$ diverges at every point of $\mathbb{R} \setminus 2\pi\mathbb{Z}$; the sequence $f_n(x) = |x|^{1+1/n}$ of differentiable functions converges uniformly on $[-1,1]$ to $f(x) = |x|$ (which is not differentiable at 0). However, the following theorem holds:

Theorem 9.1.11. *If a sequence (f_n) of differentiable functions on a bounded interval I is such that the sequence (f_n') converges uniformly on I to a function g and there is a point $x_0 \in I$ such that the sequence $(f_n(x_0))$ converges, then (f_n) converges uniformly on I to a differentiable function f and $f' = g$.*

If we assume additionally that the functions f_n' are integrable and function g has a primitive (this is so, for example, if f_n' are continuous), then this theorem immediately follows from Theorem 9.1.10.

Proof. Let $\varepsilon > 0$; for all n, m large enogh we have $\|f_n' - f_m'\| < \varepsilon/(2|I|)$ and $|f_n(x_0) - f_m(x_0)| < \varepsilon/2$, so for any $x, y \in I$,

$$\big|(f_n(y) - f_m(y)) - (f_n(x) - f_m(x))\big| < \varepsilon/(2|I|) \cdot |y - x| \le \varepsilon/2,$$

and so, for all $x \in I$,

$$|f_n(x) - f_m(x)| \le \big|f_n(x_0) - f_m(x_0)\big| + \big|(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))\big| < \varepsilon.$$

Hence, by the uniform Cauchy criterion, the sequence (f_n) converges uniformly on I to a function f.

Fix $a \in I$. The sequence $\big(\frac{f_n(x)-f_n(a)}{x-a}\big)$ converges to function $\frac{f(x)-f(a)}{x-a}$ pointwise on $I \setminus \{a\}$. For any $\varepsilon > 0$, for any n, m large enough so that $\|f_n' - f_m'\| < \varepsilon$, for all $x \in I \setminus \{a\}$ we have $\big|\frac{f_n(x)-f_n(a)}{x-a} - \frac{f_m(x)-f_m(a)}{x-a}\big| < \varepsilon$; hence, this convergence is uniform. Now, for any $x \in I \setminus \{a\}$ and any n,

$$\Big|\frac{f(x)-f(a)}{x-a} - g(a)\Big| \le \Big|\frac{f(x)-f(a)}{x-a} - \frac{f_n(x)-f_n(a)}{x-a}\Big| + \Big|\frac{f_n(x)-f_n(a)}{x-a} - f_n'(a)\Big| + |f_n'(a) - g(a)|.$$

Given $\varepsilon > 0$, there is n such that both $\big|\frac{f(x)-f(a)}{x-a} - \frac{f_n(x)-f_n(a)}{x-a}\big| < \varepsilon/3$ and $|f_n'(a) - g(a)| < \varepsilon/3$. For this n, for all $x \in I \setminus \{a\}$ close enough to a we also have $\big|\frac{f_n(x)-f_n(a)}{x-a} - f_n'(a)\big| < \varepsilon/3$. So, for all such x, $\big|\frac{f(x)-f(a)}{x-a} - g(a)\big| < \varepsilon$. This prove that $f'(a) = g(a)$. ■

A sequence of functions $f_n\colon A \longrightarrow \mathbb{R}$ is said to converge *locally uniformly* on A if every point $x \in A$ has a neighborhood J such that (f_n) converges uniformly on $J \cap A$. Since continuity, local integrability, and differentiablity are local properties, the theorems above can be adapted to local uniform convergence:

Theorem 9.1.12. (i) *If a sequence (f_n) of functions converges to a function f locally uniformly on a set A and all f_n are continuous on A, then $f|_A$ is also continuous on A.*
(ii) *If a sequence (f_n) of functions locally integrable on an interval I converges to a function f locally uniformly on I, then f is locally integrable on I, and for any $a, b \in I$, $\int_a^b f_n \longrightarrow \int_a^b f$.*
(iii) *If a sequence (f_n) of functions continuous on an interval I converges to a function f locally uniformly on I, and a sequence (F_n) of primitives of f_n converges at at least one point of I, then the sequence (F_n) converges locally uniformly on I to a differentiable function F with $F' = f$.*
(iv) *If a sequence (f_n) of differentiable functions on an interval I converges at at least one point of I and the sequence (f_n') converges to a function g locally uniformly on I, then (f_n) converges locally uniformly on I to a differentiable function f with $f' = g$.*

Let me make a small (but important) digression. I used the word "local" in two senses: I said that a function is "locally integrable on I" if it was integrable on any closed bounded subinterval of I; and now I say that a functional sequence "converges locally uniformly on I" if it converges uniformly on a neighborhood of every point of I. Actually, usually (and in these two cases in particular) these two notions of "local goodness" coincide: every point of an interval I has a bounded neighborhood whose closure, a closed bounded interval, is in I, so if we know that some property (boundedness, uniform continuity, etc.) holds for all closed bounded intervals, it holds for this neighborhood. The converse implication follows from the following fundamental theorem, which says that "any open cover of a closed bounded interval has a finite subcover":

Theorem 9.1.13. *Let I be a closed bounded interval and let $\mathcal{I}$ be a set of open intervals such that $\bigcup_{J\in\mathcal{I}} J \supseteq I$. Then there are $J_1, \dots, J_k \in \mathcal{I}$ such that $\bigcup_{i=1}^k J_i \supseteq I$.*

Proof. Assume that the assertion doesn't hold: I is not covered by a finite subset of $\mathcal{I}$. Subdivide I into two closed subintervals of length $|I|/2$; then at least one of them is not covered by a finite subset of $\mathcal{I}$; call it I_1. Subdivide I_1 into two closed subintervals of length $|I|/4$; then at least one of them is not covered by a finite subset of $\mathcal{I}$; call it I_2. Proceeding by induction, we construct a nested sequence $I \supseteq I_1 \supseteq I_2 \supseteq \cdots$ of closed intervals with $|I_n| \longrightarrow 0$, such that for every n, I_n is not covered by a finite subset of $\mathcal{I}$. Let $a \in \bigcap_{n=1}^\infty I_n$. There is $J \in \mathcal{I}$ such that $a \in J$; let c be the distance from a to the nerest endpoint of J. Let n be such that $|I_n| < c$. Then $I_n \subseteq J$, which contradicts our construction. ∎

Now, if a series (f_n) converges locally uniformly on an interval I, then for any closed bounded subinterval $[a, b]$ of I, for any point $x \in [a, b]$ there is an open interval J_x containing x such that (f_n) converges uniformly on J_x. Since $\bigcup_{x\in[a,b]} J_x \supseteq [a, b]$, there are $x_1, \dots, x_k \in [a, b]$ such that $\bigcup_{i=1}^k J_{x_i} \supseteq [a, b]$. Since (clearly) (f_n) converges uniformly on the finite union $\bigcup_{i=1}^k J_{x_i}$, it converges uniformly on $[a, b]$. Same argument applies to local boundedness, local integrability, etc.

9.2. Uniform convergence of functional series

A functional series is a formal infinite sum $f_1 + f_2 + f_3 + \dots$, or $\sum f_i$, where (f_i) is a sequence of functions on a set $A \subseteq \mathbb{R}$. A series $\sum f_i$ *converges to f pointwise on A* if $f(x) = \sum_{i=1}^\infty f_i(x)$ for all $x \in A$, or, equivalently, if the sequence of partial sums $s_n = \sum_{i=1}^n f_i$ converges to f pointwise on A. A series $\sum_{i=1}^\infty f_i$ *converges to f uniformly on A* if $s_n \rightrightarrows f$ on A. In both cases we write $f = \sum_{i=1}^\infty f_i$.

Most theorems below follow from the corresponding theorems for functional sequences.

Cauchy's criterion for uniform convergence of functional series is

Theorem 9.2.1. *A series $\sum f_i$ converges uniformly on a set A iff for any $\varepsilon > 0$ there exists k such that for any $n > m \geq k$, $\left\|\sum_{i=m+1}^n f_i\right\| < \varepsilon$.*

This follows from Theorem 9.1.6:

Theorem 9.2.2. *If two series $\sum f_i$ and $\sum g_i$ converge uniformly on a set A, then the series $\sum(f_i + g_i)$ converges uniformly an A, and $\sum_{i=1}^\infty(f_i + g_i) = \sum_{i=1}^\infty f_i + \sum_{i=1}^\infty g_i$.*

This follows from Theorems 9.1.7 and 9.1.8:

Theorem 9.2.3. *Let $f = \sum_{i=1}^{\infty} f_i$ uniformly on A. If f_i are continuous at a point $a \in A$, then $f|_A$ is continuous at a; if f_i are continuous on A, then $f|_A$ is continuous on A.*

This follows from Theorem 9.1.9:

Theorem 9.2.4. *If $f = \sum_{i=1}^{\infty} f_i$ uniformly on an interval $[a,b]$ and f_i are integrable on $[a,b]$, then f is also integrable and $\int_a^b f(x)dx = \sum_{i=1}^{\infty} \int_a^b f_i(x)dx$.*

This follows from Theorem 9.1.10:

Theorem 9.2.5. *If $f = \sum_{i=1}^{\infty} f_i$ uniformly on a bounded interval I, f_i are continuous on I, $F_i' = f_i$ for all i, and the series $\sum F_i(x_0)$ converges for some $x_0 \in I$, then the series $\sum F_i$ converges uniformly on I and $F = \sum_{i=1}^{\infty} F_i$ satisfies $F' = f$.*

This follows from Theorem 9.1.11:

Theorem 9.2.6. *If a series $\sum_{i=1}^{\infty} f_i = f$ on a bounded interval I, f_i are differentiable and the series $\sum f_i'$ converges uniformly on I, then $\sum_{i=1}^{\infty} f_i$ converges to f uniformly on I, f is differentiable and $f' = \sum_{i=1}^{\infty} f_i'$.*

A series $\sum f_i$ is said to converge *locally uniformly* on a set A if the sequence of its partial sums converges locally uniformly on A. The local uniform convergence is sufficent for the sum of the series to inherit the "good properties" of the summands:

Theorem 9.2.7. (i) *If $f = \sum_{i=1}^{\infty} f_i$ locally uniformly on a set A and f_i are continuous on A, then $f|_A$ is also continuous on A.*
(ii) *If $f = \sum_{i=1}^{\infty} f_i$ locally uniformly on an interval I and f_i are locally integrable on I, then f is also locally integrable on I, and for any $a, b \in I$, $\sum_{i=1}^{\infty} \int_a^b f_i = \int_a^b f$.*
(iii) *If $f = \sum_{i=1}^{\infty} f_i$ locally uniformly on an interval I f_i are continuous and F_i are primitives of f_i for all i, and the series $\sum F_i(x_0)$ converges for some $x_0 \in I$, then $F = \sum_{i=1}^{\infty} F_i$ locally uniformly on I and $F' = f$.*
(iv) *If $f = \sum_{i=1}^{\infty} f_i$ on an interval I, f_i are differentiable and the series $\sum_{i=1}^{\infty} f_i'$ converges locally uniformly on I, then $f = \sum_{i=1}^{\infty} f_i$ locally uniformly on I, f differentiable on I and $f' = \sum_{i=1}^{\infty} f_i'$.*

A series $\sum f_i$ is said to *converge absolutely* if the series $\sum |f_i|$ converges, and to *converge absolutely uniformly* on A if the series $\sum |f_i|$ converges uniformly on A.

Theorem 9.2.8. *If a series converges absolutely uniformly on A then it converges uniformly on A.*

Proof. For any $m < n$, $\left|\sum_{i=m+1}^{n} f_i\right| \le \sum_{i=m+1}^{n} |f_i|$, so $\left\|\sum_{i=m+1}^{n} f_i\right\| \le \left\|\sum_{i=m+1}^{n} |f_i|\right\|$, so if $\sum |f_i|$ satisfies the Cauchy criterion for uniform convergence, then $\sum f_i$ also does. ∎

There is a simple "numerical" test for absolute uniform convergence:

Theorem 9.2.9. *If a sequence (f_i) of functions on a set A is such that $\sum \|f_i\| < +\infty$, then the series $\sum f_i$ converges absolutely uniformly on A.*

Proof. For any $m < n$, $\left\|\sum_{i=m+1}^{n} |f_i|\right\| \le \sum_{i=m+1}^{n} \|f_i\|$, so if $\sum \|f_i\|$ converges and thus is Cauchy, then $\sum |f_i|$ satisfies the Cauchy criterion for uniform convegrence. ∎

The following easy corollary of Theorem 9.2.9 is called *Weierstrass's M-test*:

Theorem 9.2.10. *If a sequence (f_i) of functions on a set A and a sequence of real numbers (M_i) are such that $\|f_i\| \le M_i$ for all i and $\sum M_i < +\infty$, then the series $\sum f_i$ converges absolutely uniformly on A.*

Proof. If $\sum M_i < +\infty$, then $\sum \|f_i\| < +\infty$ by the comparison test. ∎

Examples. (i) The series $\sum_{n=1}^{\infty} \frac{1}{n} \sin(nx)$ converges pointwise on $\mathbb{R}$ to the 2π-periodic funciton f such that $f(0) = f(2\pi) = 0$ and $f(x) = \pi - x$ on $[0, 2\pi]$. (I won't prove this.) Since f is discontinuous, the convergence is not uniform.
(ii) The series $\sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nx)$ converges uniformly on $\mathbb{R}$ to a continuous function.
(iii) The series $\sum_{n=1}^{\infty} \frac{1}{n^3} \sin(nx)$ converges uniformly on $\mathbb{R}$ to a differentiable function f, with $f'(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx)$.

(iv) The series defining the zeta function, $\zeta(x)=\sum_{n=1}^{\infty}\frac{1}{n^x}$, converges for all $x>1$. This convergence is not uniform on $(1,+\infty)$ since ζ is unbounded near 1. However, it is locally uniform: for any $a>1$ for all n we have $\left\|\frac{1}{n^x}\right\|_{[a,+\infty)}=\frac{1}{n^a}$, and since $\sum\frac{1}{n^a}<\infty$ the series $\sum\frac{1}{n^x}$ converges absolutely unformly on $[a,+\infty)$. Hence, ζ is continuous. To check its differentiability, consider the series $\sum\left(\frac{1}{n^x}\right)'=\sum\frac{-\log n}{n^x}$. This series also converges locally uniformly on $(1,+\infty)$ (since for any $a>1$, $\sum\frac{\log n}{n^a}<\infty$); hence, ζ is differentiable, with $\zeta'(x)=-\sum_{n=1}^{\infty}\frac{\log n}{n^x}$. By induction, ζ is infinitely differentiable.
(v) Let $g(x)=\{x\}$ if $\{x\}\le\frac{1}{2}$ and $1-\{x\}$ if $\{x\}\ge\frac{1}{2}$, where $\{x\}$ is the fractional part of x. For every $n\in\mathbb{N}$, let $f_n(x)=\frac{1}{4^n}g(4^nx)$; then the function $f=\sum_{n=0}^{\infty}f_n$ is continuous but nowhere differentiable on $\mathbb{R}$.

Proof. Since $\|f_n\|=\frac{1}{4^n}$, $n\in\mathbb{N}$, the series $\sum_{n=1}^{\infty}f_n$ converges uniformly by M-test; so, f is defined and continuous.

Let $x\in\mathbb{R}$; we are going to prove that $f'(x)$ doesn't exist. Since f is periodic with period 1, we may assume that $0\le x<1$; let $x=0.c_1c_2\cdots$, with $c_n\in\{0,1,2,3\}$ for all n, be the quarternary (base 4) expansion of x. For any $m\in\mathbb{N}$, let $h_m=\frac{1}{4^m}$ if $c_m=0$ or 2 and $h_m=\frac{-1}{4^m}$ if $c_m=1$ or 3. Then for any $n\ge m$, $4^nh_m\in\mathbb{Z}$, so $g_n(x+h_m)=g_n(x)$. For any $n<m$, $\{4^nx\}=0.c_{n+1}c_{n+2}\cdots c_m\cdots$, and by the choice of h_m, $\{4^n(x+h_m)\}=0.c_{n+1}c_{n+2}\cdots(c_m\pm 1)\cdots$, and $\{4^n(x+h_m)\}<1/2$ iff $\{4^n(x)\}<1/2$ (even in the case $m=n+1$); so, $f_n(x+h_m)=x\pm h_m$. Therefore,

$$\frac{f(x+h_m)-f(x)}{h_m}=\frac{\sum_{n=1}^{\infty}(f_n(x+h_m)-f_n(x))}{h_m}=\frac{\sum_{n=1}^{m-1}(f_n(x)\pm h_m-f_n(x))}{h_m}=\sum_{n=1}^{m-1}\pm 1.$$

So, for any $m\in\mathbb{N}$, $\frac{f(x+h_m)-f(x)}{h_m}$ is an integer, which is even if m is odd and odd if m is even. The limit $\lim_{m\to\infty}\frac{f(x+h_m)-f(x)}{h_m}$ of this sequence cannot exist, so $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ doesn't exist. ■

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10. Power series and analytic functions

10.1. Power series

A power series, centered at a, is a functional series of the form $a_0+a_1(x-a)+a_2(x-a)^2+\ldots=\sum_{n=0}^{\infty}a_n(x-a)^n$.

The radius of convergence of a power series $\sum a_n(x-a)^n$ is $R=\left(\limsup\sqrt[n]{|a_n|}\right)^{-1}$. (If $\limsup\sqrt[n]{|a_n|}=0$, we assume $R=+\infty$; if $\limsup\sqrt[n]{|a_n|}=+\infty$, we assume $R=0$.)

Theorem 10.1.1. *If the radius of convergence R of a power series $\sum a_n(x-a)^n$ is positive, then the series converges absolutely locally uniformly on $(a-R,a+R)$ (on $\mathbb{R}$ if $R=+\infty$) and diverges at every x with $|x-a|>R$.*

Proof. For every $0<c<R$, $\limsup\sqrt[n]{|a_n|c^n}=c\limsup\sqrt[n]{|a_n|}=c/R<1$ ($=0$ if $R=+\infty$), so the series $\sum|a_n|c^n<+\infty$ by the root test. By the M-test, the series $\sum a_n(x-a)^n$ converges absolutely uniformly on the set $\{x:|x-a|\le c\}=[a-c,a+c]$. Every closed bounded subinterval of $(a-R,a+R)$ is contained in an interval $[a-c,a+c]$ for some $c<R$, so $\sum a_n(x-a)^n$ converges absolutely locally uniformly on $(a-R,a+R)$.

For any x with $|x-a|>R$, $\limsup\sqrt[n]{|a_n(x-a)^n|}=|x-a|\limsup\sqrt[n]{|a_n|}=|x-a|/R>1$, so $\sum a_n(x-a)^n$ diverges by the root test. ■

The interval $(a-R,a+R)$ is called *the interval of convergence* of the power series.

At the endpoints $a-R$ and $a+R$ of the interval of convergence, $\sum a_n(x-a)^n$ may converge and may diverge: $\sum\frac{x^n}{n}$ converges at -1 and diverges at 1.

The radius of convergence of a power series can be sometimes determined by just finding out where it converges: if $\sum a_n(x-a)^n$ converges at a point x_1, then $R\ge|x_1-a|$, if it diverges at a point x_2, then $R\le|x_2-a|$.

The radius of convergence of a power series can sometimes also be found using the ratio-test:

Theorem 10.1.2. *Let $\sum a_n(x-a)^n$ be a power series and assume that the limit $R=\lim_{n\to\infty}\frac{|a_n|}{|a_{n+1}|}$ exists, finite or infinite. Then R is the radius of convergence of the series.*

Proof. For any x, $\frac{|a_{n+1}|\cdot|x-a|^{n+1}}{|a_n|\cdot|x-a|^n} = \frac{|a_{n+1}|}{|a_n|}|x-a| \longrightarrow \frac{|x-a|}{R}$ (0 if $R=+\infty$, $+\infty$ if $R=0$). By the ratio test, $\sum\bigl|a_n(x-a)^n\bigr|$ converges if $|x-a|<R$ and diverges if $|x-a|>R$; so, R is the radius of convergence of $\sum a_n(x-a)^n$. ■

Examples. The radius of convergence
(i) of the series $\sum x^n$ is 1;
(ii) of $\sum n^{10}x^n$ is $\bigl(\lim \sqrt[n]{n^{10}}\bigr)^{-1} = 1$;
(iii) of $\sum 2^n x^n$ is $\bigl(\lim \sqrt[n]{2^n}\bigr)^{-1} = \frac{1}{2}$;
(iv) of $\sum 3^{-n}x^n$ is $\bigl(\lim \sqrt[n]{3^{-n}}\bigr)^{-1} = 3$;
(v) of $\sum \frac{1}{n!}x^n$ is $\lim(1/n!)/(1/(n+1)!) = \lim(n+1) = \infty$;
(vi) of $\sum n!x^n = \lim n!/(n+1)! = \lim 1/(n+1) = 0$;
(vii) of $\sum \binom{\alpha}{n}x^n$ is $\lim\bigl|\binom{\alpha}{n}/\binom{\alpha}{n+1}\bigr| = \lim\bigl|(n+1)/(\alpha-n)\bigr| = 1$.

In its interval of convergence, a power series defines an infinitely differentiable function, which can be diferentiated term-by-term:

Theorem 10.1.3. *Let the radius of convergence of a power series $\sum a_n(x-a)^n$ be $R>0$. Then the function $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ is infinitely differentiable on the interval $(a-R, a+R)$, with $f'(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}$ and $f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-a)^{n-k}$ for all k.*

Proof. For every $n \ge 1$, $\bigl(a_n(x-a)^n\bigr)' = na_n(x-a)^{n-1}$. The series $\sum_{n=1}^{\infty} na_n(x-a)^{n-1}$ is also a power series, and I claim that its radius of convergence is R. Indeed, the radius of convergence of the power series $\sum_{n=1}^{\infty} na_n(x-a)^{n-1}(x-a)$ is $\bigl(\limsup \sqrt[n]{n|a_n|}\bigr)^{-1} = \bigl(\lim \sqrt[n]{n}\cdot \limsup \sqrt[n]{|a_n|}\bigr)^{-1} = \bigl(\limsup \sqrt[n]{|a_n|}\bigr)^{-1} = R$, and $\sum_{n=1}^{\infty} na_n(x-a)^{n-1}$ has the same interval of convergence as this series.

So, the series $\sum_{n=1}^{\infty} na_n(x-a)^{n-1}$ of derivatives of $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges locally uniformly on $(a-R, a+R)$; this implies that f is differentiable and $f'(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}$. Then we use induction on k. ■

Theorem 10.1.4. *The primitive of the function $f(x) = \sum a_n(x-a)^n$, on the interval of convergence, is the function $\sum_{n=1}^{\infty} \frac{a_{n-1}}{n}(x-a)^n + C$.*

Proof. Since the series $\sum a_n(x-a)^n$ converges locally uniformly on its interval of convergence, to get a primitive it can be integrated term-by-term. ■

Power series can also be added and multiplied term-by-term:

Theorem 10.1.5. *If two power series $\sum a_n(x-a)^n$ and $\sum b_n(x-a)^n$ converge on $(a-R, a+R)$, $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ and $g(x) = \sum_{n=0}^{\infty} b_n(x-a)^n$, then $(f+g)(x) = \sum_{n=0}^{\infty}(a_n+b_n)(x-a)^n$ and $(fg)(x) = \sum_{n=0}^{\infty}\bigl(\sum_{k=0}^{n} a_k b_{n-k}\bigr)(x-a)^n$ on $(a-R, a+R)$.*

It follows that the radia of convergence of $\sum(a_n+b_n)(x-a)^n$ and $\sum_{n=0}^{\infty}\bigl(\sum_{k=0}^{n} a_k b_{n-k}\bigr)(x-a)^n$ are $\ge \min\{R_1, R_2\}$, where R_1 is the radius of convergence of $\sum a_n(x-a)^n$ and R_2 is the radius of convergence of $\sum b_n(x-a)^n$.

Proof. The first statement is trivial, for any $x \in (a-R, a+R)$, $f(x)+g(x) = \sum_{n=0}^{\infty} a_n(x-a)^n + \sum_{n=0}^{\infty} b_n(x-a)^n = \sum_{n=0}^{\infty}(a_n+b_n)(x-a)^n$.

For any $x \in (a-R, a+R)$,

$$f(x)g(x) = \sum_{n=0}^{\infty} a_n(x-a)^n \sum_{m=0}^{\infty} b_m(x-a)^m = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n(x-a)^n b_m(x-a)^m = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n b_m(x-a)^{n+m}.$$

Since the series $\sum_{n=0}^{\infty}|a_n(x-a)^n|$ and $\sum_{m=0}^{\infty}|b_m(x-a)^m|$ converge, the double series $\sum_{n=0}^{\infty}|a_n(x-a)^n|\sum_{m=0}^{\infty}|b_m(x-a)^m| = \sum_{n,m=0}^{\infty}|a_n|\cdot|b_m|\cdot|x-a|^{n+m}$ converges, so the double series $f(x)g(x) = \sum_{n,m=0}^{\infty} a_n b_m(x-a)^{n+m}$ converges absolutely, so $f(x)g(x) = \sum_{n=0}^{\infty}\bigl(\sum_{k=0}^{n} a_k b_{n-k}\bigr)(x-a)^n$. ■

Compositions and inverses of functions defined by power series are also given by power series:

Theorem 10.1.6. *If $f(x)=\sum_{n=0}^{\infty} a_n(x-a)^n$ for $x\in(a-R_1,a+R_1)$, $b=f(a)$, and $g(y)=\sum_{m=1}^{\infty} b_m(y-b)^m$ for $y\in(b-R_2,b+R_2)$, then $(g\circ f)(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ on $(a-R,a+R)$ for some $R>0$ and $c_0,c_1,\ldots\in\mathbb{R}$.*

Proof. Let's assume for simplicity (and w.l.o.g.) that $a=0$ and $b=f(0)=a_0=0$, so that $f(x)=\sum_{n=1}^{\infty} a_nx^n$, $|x|<R_1$, and $g(y)=\sum_{m=0}^{\infty} b_my^m$, $|y|<R_2$. Define $\widetilde{f}(x)=\sum_{n=1}^{\infty}|a_n||x|^n$, $x\in(-R_1,R_1)$ and $\widetilde{g}(y)=\sum_{m=0}^{\infty}|b_m||y|^m$, $y\in(-R_2,R_2)$. (These sieries converge.) The function $\widetilde{f}$ is continuous; let $0<R\le R_1$ be such that $|\widetilde{f}(x)|<R_2$ for all $x\in[0,R)$. Then for any such x,

$$\widetilde{g}(\widetilde{f}(x))=\sum_{m=0}^{\infty}|b_m|\Big(\sum_{n=1}^{\infty}|a_n||x|^n\Big)^m=\sum_{m=0}^{\infty}\sum_{n_1,\ldots,n_m=1}^{\infty}|b_m||a_{n_1}|\cdots|a_{n_m}||x|^{n_1+\cdots+n_m}.$$

Since this "multi-series" with nonnegative terms converges, the multi-series

$$g(f(x))=\sum_{m=0}^{\infty}b_m\Big(\sum_{n=1}^{\infty}a_nx^n\Big)^m=\sum_{m=0}^{\infty}\sum_{n_1,\ldots,n_m=1}^{\infty}b_ma_{n_1}\cdots a_{n_m}x^{n_1+\cdots+n_m}$$

converges absolutely. Hence, we may reorder it and get

$$g(f(x))=\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}b_m\sum_{1\le n_1,\ldots,n_m:n_1+\cdots+n_m=n}a_{n_1}\cdots a_{n_m}\Big)x^n. \tag{10.1}$$

■

Theorem 10.1.7. *Let $f(x)=\sum_{n=0}^{\infty}a_n(x-a)^n$, $|x-a|<R$, and $a_1\neq 0$. Then f is invertible in a neighborhood of a, and $f^{-1}(y)=\sum_{m=0}^{\infty}b_m(y-b)^m$ in a neighborhood of $b=f(a)$ for some $b_0,b_1,\ldots\in\mathbb{R}$.*

Proof. We have $f'(a)=a_1\neq 0$, and since f' is continuous, $f'(x)\neq 0$ in a neighborhood of a, so f is strictly monotone, and so invertible, in this neighborhood.

Let's assume for simplicity (and w.l.o.g.) that $a=0$, so that $f(x)=\sum_{n=1}^{\infty}a_nx^n$, $|x|<R$. Put $g(y)=\sum_{m=1}^{\infty}b_my^m$, with b_m to be determined. Then, if $f(x)$ is in the interval of convergence of this series, we have

$$g(f(x))=b_1a_1x+(b_1a_2+b_2a_1^2)x^2+\cdots,$$

which is equal to x if we put $b_1=a_1^{-1}$, $b_2=-a_1^{-2}b_1a_2$, and, by (10.1),

$$b_n=-a_1^{-n}\sum_{m=1}^{n-1}b_m\sum_{1\le n_1,\ldots,n_m:n_1+\cdots+n_m=n}a_{n_1}\cdots a_{n_m}$$

for $n\ge 2$.

The problem remains to show that g is defined in a neighborhood of 0, that is, that the series $\sum b_my^m$ has a nonzero radius of convergence; this requires cumbersome estimates, I don't see any simple proof, sorry. In the framework of the complex analysis this fact is however trivial: it turns out that a function is differentiable "in the complex sense" iff it is given by a power series(!), and, like in the real analysis, the inverse of a differentiable function with nonzero derivative is differentiable. ■

10.2. Taylor series

Let function f be infinitely differentiable at a point a. *The Taylor series* of f at a is the power series $\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$. The partial sums of the Taylor series of f at a are the Taylor polynomials of f at a.

Since the Taylor series of f is a power series, it either diverges for all $x\neq a$ (if $R=0$), or converges absolutely locally uniformly to an infinitely differentiable function g on the interval $(a-R,a+R)$, where R is its radius of convergence; but even if it converges, it is not necessarily true that $g=f|_{(a-R,a+R)}$. (For example, the Taylor series at 0 of the function $f(x)=e^{-1/x^2}$, $x\neq 0$, and $f(0)=0$, is $\sum_{n=0}^{\infty}0x^n$, which is $\neq f(x)$ for any $x\neq 0$.) We have that $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$ at a point x iff the Taylor polynomials $P_{a,n,f}$ of f converge to f at x as $n\longrightarrow\infty$, that is, iff $R_{a,n,f}(x)\underset{n\to\infty}{\longrightarrow}0$, where $R_{a,n,f}=f-P_{a,n,f}$ are the remainders in the Taylor formulas for f at a.

Theorem 10.2.1. *If the radius R of convergence of a series $\sum a_n(x-a)^n$ is positive, then $\sum_{n=0}^{\infty} a_n(x-a)^n$ is the Taylor series of the function $f(x)=\sum_{n=0}^{\infty} a_n(x-a)^n$, $x\in(a-R,a+R)$, that is, $a_n=\frac{f^{(n)}(a)}{n!}$ for all n.*

Proof. For any k, $f^{(k)}(x)=\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-a)^{n-k}$, and $f^{(k)}(a)=k!a_k$. ∎

Corollary 10.2.2. *If a function f is defined by a power series, then this power series is unique.*

Examples. If (and only if) the Taylor remainders $R_{a,n,f}(x)\longrightarrow 0$ as $n\longrightarrow\infty$ on an interval I, the Taylor polynomials $P_{a,n,f}(x)\longrightarrow f(x)$ on I, so f equals its Taylor series on I.
(i) $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ on $\mathbb{R}$.
(ii) $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$ on $\mathbb{R}$ and $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$ on $\mathbb{R}$.
(iii) $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ on $(-1,1)$.
(iv) Let $\alpha\in\mathbb{R}$; then $(1+x)^\alpha=\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n$ on $(-1,1)$. To see this, we need to estimate $R_{0,n,f}$ for $f(x)=(1+x)^\alpha$. For any $0<x<1$, in Lagrange's form, for some $c_n\in(0,x)$,

$$0\le R_{0,n,f}(x)=\binom{\alpha}{n+1}(1+c_n)^{\alpha-n-1}x^{n+1}\le\binom{\alpha}{n+1}x^{n+1}$$

for $n>\alpha$. We know that for $0<x<1$, $\sum\binom{\alpha}{n}x^n<+\infty$, so $\binom{\alpha}{n+1}x^{n+1}\longrightarrow 0$ as $n\longrightarrow\infty$, so $R_{0,n,f}(x)\longrightarrow 0$. For any $-1<x<0$, in Cauchy's form, for some $c_n\in(x,0)$,

$$R_{0,n,f}(x)=\binom{\alpha}{n}(\alpha-n)(1+c_n)^{\alpha-n-1}(x-c_n)^n x=\binom{\alpha}{n}(\alpha-n)\left(\tfrac{x-c_n}{1+c_n}\right)^n(1+c_n)^{\alpha-1}x.$$

It is easy to check that $\left|\frac{x-c_n}{1+c_n}\right|<|x|$ and the radius of convergence of the series $\sum\binom{\alpha}{n}(\alpha-n)x^n$ is 1, so $\binom{\alpha}{n}(\alpha-n)\left(\frac{x-c_n}{1+c_n}\right)^n\longrightarrow 0$ as $n\longrightarrow\infty$; also $|1+c_n|<|1+x|$ for all n, so the sequence $\left((1+c_n)^{\alpha-1}\right)$ is bounded. Hence, $R_{0,n,f}(x)\longrightarrow 0$.

Actually, these calculations were not necessary. The function $f(x)=(1+x)^\alpha=e^{\alpha\log(1+x)}$ is a composition of the function $\alpha\log(1+x)$, given by a power series in a neighborhood of 0, and $\exp y$, also given by a power series centered at $0=\alpha\log(1+0)$. Hence, f itself is given by a power series in a neighborhood of 0 by Theorem 10.1.6.
(v) $\log(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$ on $(-1,1)$, since $R_{0,n,\log(1+x)}\longrightarrow 0$ as $n\longrightarrow\infty$, or since $\log(1+x)$ is a primitive of $\frac{1}{1+x}$.
(vi) The functions $f(x)=e^{-1/x^2}$, $x\ne 0$, $f(0)=0$, is infinitely differentiable on $\mathbb{R}$ with $f^{(n)}(0)=0$ for all n, so the Taylor series of f is 0, and is not equal to f in any neighborhood of 0.

10.3. Analytic functions

Functions, locally defined by power series, are called *analytic*: A function f on a set A is said to be *analytic* on A if for every $a\in A$, in a neighborhood of a, f is representable by a power series centered at a, or, equivalently, if f is infinitely differentiable on A and for any $a\in A$ the Taylor series of f centered at a converges to f in a neighborhood of a.

Examples. (i) $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ on $\mathbb{R}$. So for any $a\in\mathbb{R}$,

$$e^x=e^{a+(x-a)}=e^a e^{x-a}=e^a\sum_{n=0}^{\infty}\frac{(x-a)^n}{n!}$$

for all $x\in\mathbb{R}$. Hence, exp is analytic on $\mathbb{R}$.
(ii) $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$ on $\mathbb{R}$ and $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$ on $\mathbb{R}$. For any $a\in\mathbb{R}$,

$$\begin{aligned}\sin x&=\sin(a+(x-a))=\sin a\cos(x-a)+\cos a\sin(x-a)\\&=\sin a\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x-a)^{2n}+\cos a\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(x-a)^{2n+1}\end{aligned}$$

for all $x\in\mathbb{R}$. So, sin (and cos) are analytic on $\mathbb{R}$.

(iii) $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ on $(-1,1)$. Thus for any $a<1$,

$$\tfrac{1}{1-x}=\tfrac{1}{(1-a)-(x-a)}=\tfrac{1}{1-a}\cdot\tfrac{1}{1-\frac{x-a}{1-a}}=\tfrac{1}{1-a}\sum_{n=0}^{\infty}\big(\tfrac{x-a}{1-a}\big)^n=\sum_{n=0}^{\infty}\tfrac{1}{(1-a)^{n+1}}(x-a)^n,$$

which converges for all x with $|x-a|<|1-a|$. So, $\frac{1}{1-x}$ is analytic on $(-\infty,1)$.
(iv) Let $\alpha\in\mathbb{R}$. We have $(1+x)^\alpha=\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n$ on $(-1,1)$. So for any $a>0$,

$$x^\alpha=(a+(x-a))^\alpha=a^\alpha\big(1+\tfrac{x-a}{a}\big)^\alpha a^\alpha\sum_{n=0}^{\infty}\tbinom{\alpha}{n}\big(\tfrac{x-a}{a}\big)^n$$

for any x with $|x-a|<a$. So, x^α is analytic on $(0,+\infty)$.
(v) We have $\log(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$ on $(-1,1)$. So for any $a>0$,

$$\log x=\log(a+(x-a))=\log a+\log\big(1+\tfrac{x-a}{a}\big)=\log a+\sum_{n=1}^{\infty}\tfrac{(-1)^{n-1}((x-a)/a)^n}{n}=\log a+\sum_{n=1}^{\infty}\tfrac{(-1)^{n-1}}{na^n}(x-a)^n$$

for any x with $|x-a|<a$. So, log is analytic on $(0,+\infty)$.
(vi) The functions $f(x)=e^{-1/x^2}$, $x\neq 0$, $f(0)=0$, is infinitely differentiable but not analytic on $\mathbb{R}$. (But is analytic on $\mathbb{R}\setminus\{0\}$).

By Theorem 10.1.3, analytic functions are infinitely differentiable. By Theorems 10.1.3–10.1.7 derivatives, primitives, sums, products, quotients, compositions and inverses of analytic functions are analytic. Since exponential, power, logarithmic, trigonometric functions are analytic, all elementary functions are analytic.

The following fact is not trivial(!):

Theorem 10.3.1. *If function f is defined by a power series on an open interval I, then f is analytic on I.*

Proof. Let $f(x)=\sum_{n=0}^{\infty}a_n(x-a)^n$ on $I=(a-R,a+R)$, or $(-\infty,+\infty)$. Let $b\in I$; we need to show that f is given by a power series in a neighborhood of b. W.l.o.g. assume that $b>a$. The series $\sum_{n=0}^{\infty}|a_n|\cdot|x-a|^n$ converges for all $x\in I$. Let $c>b$ be the right endpoint of I; for any $x\in[b,c)$ we have

$$\infty>\sum_{n=0}^{\infty}|a_n|\cdot|x-a|^n=\sum_{n=0}^{\infty}|a_n|(x-b+b-a)^n=\sum_{n=0}^{\infty}|a_n|\sum_{i=0}^{n}\tbinom{n}{i}(x-b)^i(b-a)^{n-i},$$

which is a sum of the double series $\sum_{\substack{0\leq n\\ 0\leq i\leq n}}|a_n|\binom{n}{i}(x-b)^i(b-a)^{n-i}$ with nonnegative terms. Hence, the double series $\sum_{\substack{0\leq n\\ 0\leq i\leq n}}a_n\binom{n}{i}(x-b)^i(b-a)^{n-i}$ converges absolutely for any x such that $|x-b|<r$ where $r=c-b$, and the sum doesn't depend on the order of summation. So, on one hand,

$$\sum_{\substack{0\leq n\\ 0\leq i\leq n}}a_n\tbinom{n}{i}(x-b)^i(b-a)^{n-i}=\sum_{n=0}^{\infty}a_n\sum_{i=0}^{n}\tbinom{n}{i}(x-b)^i(b-a)^{n-i}=\sum_{n=0}^{\infty}a_n\big((x-b)+(b-a)\big)^n=\sum_{n=0}^{\infty}a_n(x-a)^n$$
$$=f(x),$$

and on the other hand,

$$\sum_{\substack{0\leq n\\ 0\leq i\leq n}}a_n\tbinom{n}{i}(x-b)^i(b-a)^{n-i}=\sum_{i=0}^{\infty}\Big(\sum_{n=i}^{\infty}a_n\tbinom{n}{i}(b-a)^{n-i}\Big)(x-b)^i.$$

Hence, f is given by a power series $\sum b_n(x-b)^n$ in the neighborhood $(b-r,b+r)$ of b. ■

Recall that a point a of a set $A\subseteq\mathbb{R}$ is said to be isolated in A if a neighborhood of a contains no other points of A, $(a-\varepsilon,a+\varepsilon)\cap A=\{a\}$, and a set A is said to be discrete if all its point are isolated.

Analytic functions have the property that all their zeroes are isolated:

Theorem 10.3.2. *If a nonzero function f is analytic on an interval I, then the set $Z=\{x\in I:f(x)=0\}$ of zeroes of f is discrete.*

Proof. Let $a\in Z$, then $f(x)=\sum_{n=0}^{\infty}a_n(x-a)^n$ in a neighborhood of a. Consider two cases.

(Case 1) There are k such that $a_k \neq 0$; let k be the minimal such integer. Then $f(x) = (x-a)^k\big(a_k + a_{k+1}(x-a) + \cdots\big)$. The function $g(x) = a_k + a_{k+1}(x-a) + \cdots$ is continuous and $g(a) = a_k \neq 0$, so $g(x) \neq 0$ for all x in a neighborhood of a, and so $f(x) = (x-a)^k g(x) \neq 0$ for all $x \neq a$ in this neighborhood. Hence, a is an isolated element of Z.
(Case 2) $a_n = 0$ for all n; then $f = 0$ in a neighborhood of a. So, Z contains an interval $[a, z)$ for some $z > a$. Let $c = \sup\big\{z \in I : [a, z) \in Z\big\}$. If c is an inner point (not the right endpoint) of I, then, since c is a limit point of the set $\{x : f(x) = 0\}$, we have $f^{(n)}(c) = \lim_{x\to c^-} f^{(n)}(x) = 0$ for all n, so $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = 0$ in a neighborhood $(c-\delta, c+\delta)$, with $\delta > 0$, so $f = 0$ on $[a, c+\delta)$, which contradicts the choice of c. So, $f(x) = 0$ for all $x \in I$ with $x \geq a$. Similarly, $f(x) = 0$ for all $x \in I$ with $x \leq a$. Hence, $f = 0$ on I. ■

Theorem on isolated zeroes implies the following property of *rigidity* of analytic functions:

Theorem 10.3.3. *If two functions f and g are analytic on an interval I and coincide on a subset of I that has a limit point in I, then $f = g$ on I.*

Proof. Let $A \subseteq I$, let $a \in I$ be a limit point of A, and assume that $f|_A = g|_A$. Define $h = f - g$; then h is analytic on I and $h = 0$ on A. By continuity, $h(a) = 0$ as well, so a is a non-isolated zero of h. Hence, $h = 0$, and so, $f = g$ on I. ■

Note that *smooth* (that is, infinitely differentiable) functions are not, in general, rigid: the function $\begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$ is infinitely differentiable and is equal to 0 on $(-\infty, 0]$, but is not identically zero.

10.4. Abel's theorem

Abel's theorem says that if a power series converges at an endpoint of its interval of convergence, then it converges to "the right" sum:

Theorem 10.4.1. *Let $0 < R < +\infty$ be the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, and assume that the series converges at $x = a + R$ as well. Then the convergence is uniform on $[a, a+R]$, and so, the function $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ is continuous on $(a-R, a+R]$.*

Proof. W.l.o.g., we may assume that $a = 0$ and $R = 1$. Let $\varepsilon > 0$; since the series $\sum a_n$ converges, by Cauchy's criterion, there exists k such that for any $m \geq k$ all the sums $s_i = \sum_{n=m+1}^{m+i} a_n$, $i \geq 1$, satisfy $|s_i| < \varepsilon$. Let $m \geq k$, $d \geq 1$; for any $x \in [0, 1)$ by the summation by parts formula (Lemma 8.3.2),

$$\sum_{n=m+1}^{m+d} a_n x^n = x^{m+1}\sum_{i=1}^{d} a_{m+i}x^{i-1} = x^{m+1}\Big(s_d x^{d-1} - \sum_{i=1}^{d-1} s_i(x^i - x^{i-1})\Big) = s_d x^{m+d} + x^{m+1}(1-x)\sum_{i=1}^{d-1} s_i x^{i-1}.$$

Now, for any $x \in (0, 1)$, $|s_d x^{m+d}| \leq |s_d| < \varepsilon$ and

$$\Big|x^{m+1}(1-x)\sum_{i=1}^{d-1} s_i x^{i-1}\Big| \leq (1-x)\sum_{i=1}^{d-1}|s_i|x^{i-1} \leq \varepsilon(1-x)\sum_{i=1}^{d-1} x^{i-1} \leq \varepsilon(1-x)\frac{1-x^{d-1}}{1-x} = \varepsilon(1-x^{d-1}) < \varepsilon.$$

So, $\big|\sum_{n=m+1}^{m+d} a_n x^n\big| < 2\varepsilon$ for all $x \in [0, 1)$, and also for $x = 1$. Hence, by the uniform Cauchy criterion, the series $\sum a_n x^n$ converges uniformly on $[0, 1]$, so the function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is continuous on $[0, 1]$. ■

As a corollary, we obtain:

Theorem 10.4.2. *If a series $\sum a_n$ converges, then $f(1) = \sum_{n=0}^{\infty} a_n = \lim_{x\to 1^-} f(x)$, where $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $x \in (0, 1]$.*

Proof. If $\sum a_n$ converges, then the series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = 1$, so the radius R of convergence of this power series is ≥ 1, so, whether $R > 1$ or $= 1$, f is left-continuous at 1. ■

For a series $\sum_{n=0}^{\infty} a_n$, the function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is called *the generating function*. If $\sum a_n$ converges, then f is defined and analytic on $(-1, 1)$, and continuous on $(-1, 1]$. It may however be that $\sum a_n$ diverges, but f is defined on $(-1, 1)$ and is extendible by continuity to 1. In this case, $f(1)$ is called *the Abel sum* of $\sum_{n=0}^{\infty} a_n$.

Examples. (i) The series $\sum \frac{(-1)^{n-1}}{n}$ converges (by the alternating series test), and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n} = \log(1+x)$ for $|x| < 1$, so $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log(1+1) = \log 2$.
(ii) The series $1 - \frac{1}{3} + \frac{1}{5} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges, and $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \arctan x$ for $|x| < 1$, so $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \arctan 1 = \pi/4$.
(iii) The series $1 - 1 + 1 - 1 + \cdots = \sum_{n=0}^{\infty}(-1)^n$ diverges. Its generating function is $\sum_{n=0}^{\infty}(-1)^n x^n = \frac{1}{1+x}$. This function $\frac{1}{1+x}$ is defined at 1 and equals $\frac{1}{2}$; so, the Abel sum of $\sum_{n=0}^{\infty}(-1)^n$ is $\frac{1}{2}$.