Math 4181H **Homework 1**

Due by Tuesday, September 2

Please turn in your solutions through Carmen; however, if for some reason you are having trouble with this, you can email them to me or hand them over to me in class.

In this homework you'll need to justify some elementary algebraic identities, inequalities, and transformations; in all further homeworks you will be allowed to use these identities without justification. You don't have to explicitly state the axioms you use, but it should be clear that your proofs are based on the axioms (P1)-(P13) and/or theorems from class or lecture notes.

Chapter 1, pp. 13-18:

1. Prove the following:
(ii) For any $x, y \in \mathbb{R}$, $x^2 - y^2 = (x-y)(x+y)$.
(iii) If $x^2 = y^2$, then $x = y$ or $x = -y$.

Cf. 2. What is wrong with the following proofs?
(a) *Let* $x = y \neq 0$. *Then* $x^2 = xy$, *so* $x^2 - y^2 = xy - y^2$, *so* $(x+y)(x-y) = y(x-y)$, *so* $x + y = y$, *so* $2y = y$, *so* $2 = 1$.
(b) *If* $2 = 1$ *then* $2 \cdot 2 = 2 \cdot 1$ *then* $4 = 2$ *then* $4 - 3 = 2 - 3$ *then* $1 = -1$ *then* $1^2 = (-1)^2$ *then* $1 = 1$, *which is true. So,* $2 = 1$.

We define a/b (as well as $a : b$ and $\frac{a}{b}$) as ab^{-1}.

3. Let $a, b, c, d \in \mathbb{R}$. Prove the following:
(i) If $b, c \neq 0$, then $a/b = (ac)/(bc)$.
(ii) if $b, d \neq 0$, then $a/b + c/d = (ad + bc)/(bd)$.
(vi) If $b, d \neq 0$, then $a/b = c/d$ iff $ad = bc$.
(vii) If $a, b \neq 0$, prove that $a/b = b/a$ iff $a = b$ or $a = -b$.

For $a, b \in \mathbb{R}$ we define $a < b$ if $b - a \in P$ (that is, is positive), $a > b$ if $b < a$, $a \leq b$ if $a < b$ or $a = b$, and $a \geq b$ if $b \leq a$.

5. Let $a, b, c, d \in \mathbb{R}$. Prove the following:
(i) If $a < b$ and $c < d$ then $a + c < b + d$.
(iii) If $a < b$ and $c > d$ then $a - c < b - d$.
(v) If $a < b$ and $c < 0$ then $ac > bc$.
(vi) If $a > 1$ then $a^2 > a$.
(vii) If $0 < a < 1$ then $a^2 < a$.
(viii) If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.
(ix) If $0 \leq a < b$ then $a^2 < b^2$.
(x) If $a, b \geq 0$ and $a^2 < b^2$, then $a < b$.

8. Let F be a field (that is, a set with two operations satisfying (P1)–(P9)) with an order "$<$" satisfying (P10′)–(P12′) (from my Lecture Notes) or (P′10)-(P′13) from exercise 8 in Spivak. Define $P = \{a \in F : a > 0\}$. Prove that P satisfies (P10)–(P12).

12. Recall that, for $x \in \mathbb{R}$, the absolute value $|x|$ is defined by $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. Prove the following:
(ii) For any $x \in \mathbb{R} \setminus \{0\}$, $|1/x| = 1/|x|$.
(iii) For any $x \in \mathbb{R}$ and $y \in \mathbb{R} \setminus \{0\}$, $|x|/|y| = |x/y|$.
(iv) For any $x, y \in \mathbb{R}$, $|x - y| \leq |x| + |y|$.
(v) For any $x, y \in \mathbb{R}$, $|x| - |y| \leq |x - y|$.
(vi) For any $x, y \in \mathbb{R}$, $\big||x| - |y|\big| \leq |x - y|$.

20. Prove that if $x, x_0, y, y_0, \varepsilon$ are real numbers such that $|x - x_0| < \varepsilon/2$ and $|y - y_0| < \varepsilon/2$, then $|(x+y) - (x_0 + y_0)| < \varepsilon$ and $|(x-y) - (x_0 - y_0)| < \varepsilon$.