Math 4181H **Homework 10**

Due by Tuesday, December 9

15pt **A1.** For each of the following series determine whether it converges absolutely, converges conditionally, diverges to ∞, or just diverges (and name the test that proves your claim):
(a) $\sum \frac{(-1)^n}{n^{1/n}}$; (b) $\sum \frac{(-1)^n n^2}{2^n}$; (c) $\sum \frac{(-1)^n}{\sqrt{n}}$; (d) $\sum \frac{\sin n}{\log n}$; (e) $\sum \sin(1/n)$.

5pt **A2.** Prove that the series $\sum \frac{1}{n(\log n)(\log(\log n))^\alpha}$ converges for $\alpha > 1$ and diverges for $0 < \alpha \leq 1$.

10pt **A3.** If Λ is an uncountable set, $\Lambda \longrightarrow \mathbb{R}$, $i \mapsto a_i$, is a mapping, and $a_i > 0$ for all i, prove that $\sum a_i = \infty$. (That is, prove that for any $M \in \mathbb{R}$ there exists a finite set $F \subset \Lambda$ such that $\sum_{i \in F} a_i > M$.) (*Hint:* Can the sets $\Lambda_n = \{i \in \Lambda : a_i \geq 1/n\}$ be all finite?)

Chapter 23, pp. 489-498:

10pt **10.** If $a_i = b_i = \frac{(-1)^i}{\sqrt{i}}$, $i \in \mathbb{N}$, prove that the series $\sum a_i$ (and $\sum b_i$) converge but their Cauchy product $\sum_{k=2}^{\infty}\big(\sum_{i=1}^{k-1} a_i b_{k-i}\big)$ diverges.

10pt **A4.** A function h on an interval $[a, b]$ is said to be *a step function* if there is a partition $a = x_0 < x_1 < \ldots < x_n = b$ of $[a, b]$ such that for every i, h is constant on the interval (x_{i-1}, x_i). Prove that for any closed bounded interval $[a, b]$ the step functions are dense in $C([a, b])$ with respect to the uniform norm: for every continuous function f on $[a, b]$ and any $\varepsilon > 0$ there exists a step function h on $[a, b]$ such that $\|f - h\| < \varepsilon$.

A5. Let (f_n) be a sequence of functions $A \longrightarrow \mathbb{R}$ and let $f_n \rightrightarrows f$.

5pt (a) If all f_n are uniformly continuous on A prove that f is uniformly continuous on A.

5pt (b) If the sequence (f_n) is "uniformly Lipschitz on A", that is, if there is C such that $|f_n(x) - f_n(y)| \leq C|x - y|$ for all $n \in \mathbb{N}$ and all $x, y \in A$, prove that f is Lipschitz on A. (Actually, pointwise convergence $f_n \longrightarrow f$ suffices for this.)

Chapter 24, pp. 517-525:

2. Find the pointwise limit of (f_n) and decide whether (f_n) converges uniformly to this limit.

5pt (i) $f_n(x) = x^n - x^{2n}$ on $[0, 1]$.

5pt (ii) $f_n(x) = \frac{nx}{1+n+x}$ on $[0, \infty)$.

5pt (iv) $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$ on $\mathbb{R}$.

10pt **29.** (a) Suppose that (f_n) is a sequence of continuous functions on $[a, b]$ that converges uniformly on $[a, b]$ to a function f, and let (x_n) be a sequence in $[a, b]$ with $\lim x_n = c$. Prove that $\lim f_n(x_n) = f(c)$.

5pt (b) Is this statement true without assuming that f_n are continuous?