**Math 4181H** **Homework 2**

Due by Tuesday, September 9

15pt **A1.** Let $a, b \in \mathbb{R}$, $0 < a < b$; let $A = \frac{a+b}{2}$ (the airthmetic mean of $a$ and $b$), $G = \sqrt{ab}$ (the geometric mean), $H = \left(\frac{a^{-1}+b^{-1}}{2}\right)^{-1}$ (the harmonic mean), and $Q = \sqrt{\frac{a^2+b^2}{2}}$ (the quadratic mean). Prove that $a < H < G < A < Q < b$.

5pt **A2.** (a) Prove that for any $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{R}$, $\left|\sum_{i=1}^n a_i\right| \leq \sum_{i=1}^n |a_i|$. (*Hint:* Use induction on $n$.)

10pt (b) Prove the following version of the Cauchy-Schwarz inequality: for any $n \in \mathbb{N}$ and $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$, $\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) \geq \left(\sum_{i=1}^n a_i b_i\right)^2$, with "=" only if $a_1 = \cdots = a_n = 0$ or $b_1 = \cdots = b_n = 0$ or there is $x \in \mathbb{R}$ such that $b_1 = a_1 x$, ..., $b_n = a_n x$. (*Hint:* The same proof as we used for $n = 2$ works for any $n$.)

5pt (c) Prove the $n$-dimensional triangle inequality: for any $n \in \mathbb{N}$ and $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$, $\sqrt{\sum_{i=1}^n (a_i + b_i)^2} \leq \sqrt{\sum_{i=1}^n a_i^2} + \sqrt{\sum_{i=1}^n b_i^2}$ with "=" only if $a_1 = \cdots = a_n = 0$ or $b_1 = \cdots = b_n = 0$ or there is $x \in \mathbb{R}$ such that $b_1 = a_1 x$, ..., $b_n = a_n x$.

Chapter 2, pp. 27-33:

**1(ii).** Prove by induction:

10pt (i) For all $n$, $1^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

10pt (ii) For all $n$, $1^3 + \ldots + n^3 = (1 + \ldots + n)^2$.

**Cf. 23.** Prove (by induction) that for any $a, b \in \mathbb{R}$ and $n, m \in \mathbb{N}$

5pt (i) $(a^n)^m = a^{nm}$,

5pt (ii) $(ab)^n = a^n b^n$,

5pt (iii) $(a^{-1})^n = (a^n)^{-1}$. (*Hint:* Here you don't need induction, just use (ii).)

5pt **A3.** (a) Prove that for all integer $n \geq 3$, $2^n > 2n + 1$.

5pt (b) Prove that for all integer $n \geq 5$, $2^n > n^2$.