Math 4181H **Homework 3**

Due by Tuesday, September 16

The number of elements in a (finite) set X is called *the cardinality* of X and is denoted by $|X|$ or $\#X$.

10pt **A1.** Prove that for any $n, k \in \mathbb{N}$ with $k \leq n$, $\binom{n}{k}$ equals the number of k-element subsets in an n-element set: if X is a set with $|X| = n$, then $\binom{n}{k} = \#\{A \subseteq X : |A| = k\}$. (*Hint:* Use induction on n: assume that the statement is true for some n and let X be an $(n+1)$-element set. Pick an element x_0 of X; then k-element subsets of X are of two sorts: those that contain x_0 (how many of such are there?) and those that are subsets of $X \setminus \{x_0\}$ (how many of such are there?).)

Chapter 2, pp. 27-33:

10pt **3.** (e) Prove that for any $n \in \mathbb{N}$:

(i) $\sum_{j=0}^{n} \binom{n}{j} = 2^n$. (*Hint:* No induction is needed, you may just use the binomial formula with suitable a, b.)

(ii) $\sum_{j=0}^{n} (-1)^j \binom{n}{j} = 0$. (*Hint:* No induction is needed, you may just use the binomial formula.)

(iii, iv) $\sum_{\substack{0 \leq j \leq n \\ j \text{ is odd}}} \binom{n}{j} = 2^{n-1}$ and $\sum_{\substack{0 \leq j \leq n \\ j \text{ is even}}} \binom{n}{j} = 2^{n-1}$. (*Hint:* Use (i) and (ii).)

10pt **A2.** Prove that for any $m \in \mathbb{N}$ and $n \in \mathbb{Z}$ there exist $k, r \in \mathbb{Z}$ with $0 \leq r < m$ such that $n = km + r$. (*Hint:* You may start with the case $n \in \mathbb{N}$ and use (complete) induction.)

5pt **A3.** (b) Show (by example) that if $a, b \in \mathbb{R}$ are irrational, then $a + b$ can be rational or irrational. Prove that if a is irrational and b is rational, then $a + b$ is irrational.

5pt (c) Show (by example) that if $a, b \in \mathbb{R}$ are irrational, then ab can be rational or irrational. Prove that if a is irrational and $b \neq 0$ is rational, then ab is irrational.

5pt (a) If $a \in \mathbb{R}$ is irrational, prove that a^{-1} is irrational.

5pt (d) Prove that if $a > 0$ is irrational, then $\sqrt{a}$ is irrational. Show (by example) that if $a \in \mathbb{R}$ is irrational, then a^2 can be rational or irrational.

5pt **14(b).** Prove that $\alpha = \sqrt{2} + \sqrt{3}$ is irrational. (*Hint:* Consider α^2.)

5pt **A4.** Prove that the set of irrational numbers is dense in $\mathbb{R}$. (*Hint:* Consider the set $\sqrt{2} + \mathbb{Q}$.)

5pt **A5.** Let $A = \{a \in \mathbb{Q} \mid a^2 < 2\}$. Find $\sup A$ and $\inf A$ (and prove your statement, of course).

5pt **A6.** (a) For a nonempty set $A \subseteq \mathbb{R}$ and a number $c \in \mathbb{R}$ define $cA = \{ca,\ a \in A\}$. Prove that if $c > 0$, then $\sup(cA) = c \sup A$, and if $c < 0$, then $\sup cA = c \inf A$. (You can consider the cases where A is bounded/unbounded above separately, but you can also try to unite these two cases by using "the criterion" for sup/inf.)

10pt (b) Let A and B be nonempty subsets of $(0, +\infty)$ (that is, $a, b > 0$ for all $a \in A$ and $b \in B$). Let $AB = \{ab : a \in A,\ b \in B\}$. Prove that $\sup(AB) = \sup A \sup B$.