Math 4181H **Homework 4**

Due by Tuesday, September 30

5pt **A1.** If (x_n) and (y_n) are two sequence in $\mathbb{R}$ converging to the same limit a, prove that the sequence $(x_1, y_1, x_2, y_2, \ldots)$ also converges to a.

Chapter 22, pp. 460-465:

1. Prove that

10pt (iii) $\lim\left(\sqrt[8]{n^2+1} - \sqrt[4]{n+1}\right) = 0$.

5pt (iv) $\lim(n!/n^n) = 0$. (*Hint:* Show that for all n, $n!/n^n \le 1/n$.)

5pt (vii) $\lim \sqrt[n]{n^2+n} = 1$. (*Hint:* Use the squeeze theorem.)

5pt (viii) For any $a, b \ge 0$, $\lim \sqrt[n]{a^n + b^n} = \max\{a, b\}$. (*Hint:* Assuming $a \ge b$, notice that $1+b^n/a^n \le 2$ for all n.)

2. Find the following limits

5pt (ii) $\lim\left(n - \sqrt{n+a}\sqrt{n+b}\right)$.

5pt (v) $\lim \frac{a^n - b^n}{a^n + b^n}$. (*Hint:* It is assumed that $a \neq -b$ here, since otherwise the denominator is equal to 0 for odd n. Consider the cases $a = b$, $|a| > |b|$ and $|a| < |b|$ separately.)

5pt (vii) $\lim \frac{2^{n^2}}{n!}$.

5pt **5.** (a) If $0 < a < 2$, prove that $a < \sqrt{2a} < 2$.

5pt (b) Prove that the sequence $x_1 = \sqrt{2}$, $x_2 = \sqrt{2\sqrt{2}}$, $x_3 = \sqrt{2\sqrt{2\sqrt{2}}}$, ..., converges. (*Hint:* Using (a), prove that (x_n) is increasing and bounded.)

5pt (c) Find $\lim x_n$.

6. Let $0 < a_1 < b_1$ and define $a_{n+1} = \sqrt{a_n b_n}$ and $b_{n+1} = \frac{1}{2}(a_n + b_n)$.

5pt (a) Prove that (a_n) increases, (b_n) decreases, and both converge.

5pt (b) Prove that they have the same limit.

10pt **7(b).** Find $1 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\ldots}}}$, that is: Let $x_1 = 1$ and $x_{n+1} = 1 + \frac{1}{1+x_n}$, $n \in \mathbb{N}$; prove that the sequence (x_n) converges and find its limit. (*Hint:* Prove that $|x_{n+2} - x_{n+1}| \le \frac{1}{4}|x_{n+1} - x_n|$ for all n and use Cauchy's criterion.)