Math 4181H **Homework 5**

Due by Tuesday, October 7

5pt **A1.** Let (x_n) be a sequence in $\mathbb{R}$. Prove that if (x_n) doesn't diverge to ∞, then it has at least one limit point. (*Hint:* Use the Bolzano-Weierstrass theorem.)

A2. Let (x_n) be a sequence in $\mathbb{R}$.

5pt (a) Prove that $\limsup x_n = +\infty$ iff (x_n) is unbounded above.

5pt (b) Prove that $\limsup x_n = -\infty$ iff $x_n \longrightarrow -\infty$.

5pt **A3.** Let (x_n) be a sequence in $\mathbb{R}$; prove that (x_n) has limit (finite or infinite) iff $\limsup x_n = \liminf x_n$, in which case $\lim x_n = \limsup x_n$.

10pt **A4.** Let (x_n) and (y_n) be two sequences in $\mathbb{R}$ and suppose (y_n) converges. Prove that $\limsup(x_n + y_n) = \limsup x_n + \lim y_n$.

A5. Let (x_n) be a sequence in $\mathbb{R}$.

10pt (a) Prove that $\limsup \frac{x_1+\cdots+x_n}{n} \leq \limsup x_n$. (Similarly, $\liminf \frac{x_1+\cdots+x_n}{n} \geq \liminf x_n$; don't prove it.) (*Hint:* You can try to use Stolz's theorem, or prove this directly.)

5pt (b) Prove that if $\lim x_n$ exists, then $\lim \frac{x_1+\cdots+x_n}{n}$ also exists and equals $\lim x_n$. ($\lim \frac{x_1+\cdots+x_n}{n}$ is called *the Cesáro limit* of (x_n); it may exist even if (x_n) has no conventional limit: the Cesáro limit of $(1, -1, 1, -1, \ldots)$ equals $1/2$.)

Chapter 5, pp. 108-112:

5pt **8.** (a) If a finite $\lim_{x\to a} f(x)$ and a finite $\lim_{x\to a} g(x)$ do not exist, can a finite $\lim_{x\to a}(f(x)+g(x))$ exist? Can a finite $\lim_{x\to a} f(x)g(x)$ exist?

5pt (b) If a finite $\lim_{x\to a} f(x)$ and a finite $\lim_{x\to a}(f(x)+g(x))$ exist, must a finite $\lim_{x\to a} g(x)$ exist?

5pt (c) If a finite $\lim_{x\to a} f(x)$ exists and a finite $\lim_{x\to a} g(x)$ does not exist, can a finite $\lim_{x\to a}(f(x)+g(x))$ exist?

5pt (d) If finite $\lim_{x\to a} f(x)$ and finite $\lim_{x\to a} f(x)g(x)$ exist, does it follow that $\lim_{x\to a} g(x)$ exists?

5pt **9.** Prove that $\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h)$ (that is, one limit exists iff the other exists, and they are equal). Only consider the case of finite limits.

5pt **14(a).** Prove that if $\lim_{x\to 0} f(x)/x = l \in \mathbb{R}$ and $b \neq 0$, then $\lim_{x\to 0} f(bx)/x = bl$.

10pt **34.** Prove that $\lim_{x\to 0^+} f(1/x) = \lim_{x\to+\infty} f(x)$ (that is, one limit exists iff the other exists, and they are equal). Only consider the case of finite limits.