Math 4181H **Homework 7**

Due by Tuesday, October 28

Chapter 9, pp. 163-166:

5pt **19(a).** Suppose that f, g and h are defined in a neighborhood of a, f and h are differentiable at a, $f(a) = h(a)$, $f'(a) = h'(a)$, and $f(x) \leq g(x) \leq h(x)$ or $h(x) \leq g(x) \leq f(x)$ for all x in a neighborhood of a. Prove that g is differentiable at a with $g'(a) = f'(a)$.

5pt **22.** (a) Suppose f is differentiable at a. Prove that $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a-h)}{2h}$.

5pt (b) Give an example of a function for which the limit in (a) exists and is finite, but which is not differentiable at a.

A function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is said to be *even* if $f(-x) = f(x)$ for all $x \in \mathbb{R}$ and *odd* if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

10pt **23-24.** Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable on $\mathbb{R}$. If f is even, prove that f' is odd, and if f is odd, prove that f' is even.

Chapter 10, pp. 181-187:

5pt **16.** (a) If f is differentiable at a and $f(a) \neq 0$, prove that $|f|$ is also differentiable at a.

5pt (b) Give a counterexample if $f(a) = 0$.

5pt (c) If f and g are differentiable at a and $f(a) \neq g(a)$, prove that $\max\{f, g\}$ is differentiable at a.

5pt **29.** Suppose f is differentiable at 0 and that $f(0) = 0$. Prove that $f(x) = xg(x)$ for some function g which is continuous at 0.

Chapter 11, pp. 206-211:

10pt **11.** Among all circular cylinders of volume V find the one with the smallest surface area. (*Hint:* For the cylinder of height h and base radius r the volume is $V = \pi r^2 h$ and the surface area is $S = 2\pi rh + 2\pi r^2$.)

5pt **A1.** (a) Prove that if f is convex on an open interval I and g is increasing and convex on $f(I)$, then $g \circ f$ is convex on I. (*Hint:* Use the "t-definition" of convexity: $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$.)

10pt (b) Let f be a convex function on an open interval I. Prove that if f is strictly decreasing on I, then its inverse f^{-1} is also convex on $f(I)$, and if f is strictly increasing on I, then f^{-1} is concave on $f(I)$.

Chapter 11, Appendix, p. 228:

10pt **8.** Prove Jensen's inequality: If f is a convex function on an interval I and $p_1, \ldots, p_n$ are positive numbers such that $\sum_{i=1}^n p_i = 1$, prove that for any $x_1, \ldots, x_n \in I$, $f\big(\sum_{i=1}^n p_i x_i\big) \leq \sum_{i=1}^n p_i f(x_i)$. (*Hint:* Try to apply induction on n and the "t-definition" of convexity.)

5pt **A2.** By A1, the (strictly increasing) function $\log_2$ is (strictly) concave. Use this fact and the preceding problem to prove the general arithmetic-geometric mean inequality: for any n and positive $x_1, \ldots, x_n$, $\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}$.