**Math 4181H** **Homework 8**

Due by Wednesday, November 12

5pt **A3.** Using the definition (and not using the Fundamental Theorem of Calculus), prove that $\int_0^1 x^2 = 1/3$. (*Hint:* $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.)

Chapter 13, pp. 275-281:

**4.** Let $d \in \mathbb{N}$ and $0 < a < b$, put $f(x) = x^d$. Find $\int_a^b f$ in the following way: Put $c = b/a$.

5pt (a) For each $n \in \mathbb{N}$, put $t_i = ac^{i/n}$, $i = 0, 1, \ldots, n$, and let $P_n = \{t_0, \ldots, t_n\}$. Prove that $\mathrm{mesh}(P_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

10pt (b) Find $U(f, P_n)$. (That is, try to write $U(f, P_n)$ in a compact form that will help you in (c) below.)

5pt (c) Conclude that $\int_a^b x^d dx = \frac{1}{d+1}(b^{d+1} - a^{d+1})$.

**21.** Let $f$ be strictly increasing and continuous on $[c, d]$, let $a = f(c)$ and $b = f(d)$.

10pt (a) If $P = \{t_0, \ldots, t_n\}$ is a partition of $[a, b]$, let $P' = \{f^{-1}(t_0), \ldots, f^{-1}(t_n)\}$. Prove that $L(f^{-1}, P) + U(f, P') = bd - ac$.

5pt (b) Prove that $\int_a^b f^{-1}(y)dy = bd - ac - \int_c^d f(x)\,dx$.

5pt (c) Find $\int_a^b \sqrt[n]{x}\,dx$ for $0 \le a < b$ (without using the F.T.C.)

**31.** Let $f$ be integrable on $[a, b]$.

5pt (a) Give an example where $f \ge 0$, $f(x) > 0$ for some $x \in [a, b]$, but $\int_a^b f(x)\,dx = 0$.

10pt (b) Suppose that $f \ge 0$, $f$ is continuous at $x_0 \in [a, b]$ and $f(x_0) > 0$. Prove that $\int_a^b f(x)\,dx > 0$.

Chapter 14, pp. 296-302:

**1.** Find the derivatives of the following functions:

5pt (i) $F(x) = \int_0^{x^3} \sin^3 t\,dt$. (*Hint:* $F$ is the composition of an integral function and of $x^3$.)

5pt (ii) $F(x) = \int_3^{\int_1^x \sin^3 t dt} \frac{dt}{1+\sin^6 t + t^2}$.

10pt **9.** Prove that if $f$ is continuous on $\mathbb{R}$, then $\int_0^x f(u)(x-u)\,du = \int_0^x \left(\int_0^u f(t)\,dt\right) du$. (*Hint:* Differentiate $G(x) = \int_0^x f(u)(x-u)\,du$. Or integrate $\int_0^x f(u)(x-u)\,du$ by parts.)

5pt **21.** Suppose that $f'$ is integrable on $[0, 1]$ and $f(0) = 0$. Prove that for all $x \in (0, 1]$ we have $|f(x)| \le \sqrt{\int_0^x |f'|^2 dt}$.