Math 4181H **Homework 9**

Due by Tuesday, November 25

10pt **A1.** Let $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. Prove that f is infinitely differentiable on $\mathbb{R}$ with $f^{(n)}(0) = 0$ for all n. (*Hint:* Use induction. Prove that for any polynomial p, $\lim_{x\to 0} e^{-1/x^2} p(1/x) = 0$, for which goal pass to $y = 1/x$.)

Chapter 23, pp. 489-498:

10pt **2.** Prove that the series $\sum a^n n!/n^n$ converges for $0 < a < e$ and diverges for $a > e$. (*Hint:* Use the ratio test.)

5pt **5.** (a) Prove that if the series $\sum x_i$ converges absolutely, then so does $\sum x_i^3$.

10pt (b) Show that the series $\sum_{i=1}^{\infty} x_i = 1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{\sqrt[3]{2}} - \frac{1}{2\sqrt[3]{2}} - \frac{1}{2\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} - \frac{1}{2\sqrt[3]{3}} - \frac{1}{2\sqrt[3]{3}} + \cdots$ converges, but $\sum x_i^3$ diverges.

10pt **A2.** (a) Let $f\colon [1, +\infty) \longrightarrow \mathbb{R}$ be a decreasing nonnegiative function. For every $i \in \mathbb{N}$, let $a_i = f(i)$. Prove that a finite limit $l = \lim_{n\to\infty}\big(\sum_{i=1}^{n} a_i - \int_1^n f\big)$ exists and satisfies $0 \leq l \leq a_1$. (*Hint:* Notice that for any i, $a_i \geq \int_i^{i+1} f \geq a_{i+1}$.)

5pt (b) Prove that a finite limit $\gamma = \lim_{n\to\infty}\big(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\big)$ exists. (This $\gamma = 0.5772\ldots$ is called Euler-Mascheroni constant.)