Math 4181H **Midterm 1**

In your solutions you may use axioms and all results proven in class, in homework, in lecture notes, or in the list of "review problems".

10% **1.** If $a, b \in \mathbb{R}$ and $b < a + 1/n$ for all $n \in \mathbb{N}$, prove that $b \leq a$.

10% **2.** (a) Let A and B be nonempty sets of real numbers such that $a < b$ for all $a \in A$ and all $b \in B$. Prove that there is $x \in \mathbb{R}$ such that $a \leq x \leq b$ for all $a \in A$ and all $b \in B$.

10% (b) Give an example of nonepty sets A and B of rational numbers such that $a < b$ for all $a \in A$ and all $b \in B$ and there are no $x \in \mathbb{Q}$ such that $a \leq x \leq b$ for all $a \in A$ and all $b \in B$. (And justify your answer, of course.)

15% **3.** Let $A \subseteq \mathbb{R}$ be bounded above and have no maximal element. Prove that $\sup A$ is a limit point of A.

15% **4.** Let $a \in \mathbb{R} \setminus \{0\}$ and let $\varepsilon > 0$. If $|x - a| < \min\big\{|a|, \varepsilon/(3|a|)\big\}$, prove that $|x^2 - a^2| < \varepsilon$.

15% **5.** Let $a > 1$, let $n, m \in \mathbb{N}$, $n < m$, and assume that $b = \sqrt[n]{a}$ and $c = \sqrt[m]{a}$ exist (so that $b^n = c^m = a$). Prove that $b > c$.

10% **6.** Prove that for every even $n \in \mathbb{N}$ there exist $r \in \mathbb{N}$ and an odd $m \in \mathbb{N}$ such that $n = 2^r m$.

10% **7.** (a) Prove that the set of transcendental (that is, non-algebraic) numbers is dense in $\mathbb{R}$.

10% (b) Prove that Cantor's set contains a transcendental number.