Math 4181H **Final exam review problems**

1. Let $\sum a_i$ be a converging series. Prove or disprove:
(i) If $b_i \longrightarrow 0$, then the series $\sum a_i b_i$ converges.
(ii) If $b_i \longrightarrow 0$ and $b_i \geq 0$ for all i, then $\sum a_i b_i$ converges.
(iii) If $b_i \searrow 0$ (decreases and tends to 0), then $\sum a_i b_i$ converges.
(iv) If $\sum a_i$ converges absolutely and $b_i \longrightarrow 0$, then $\sum a_i b_i$ converges.

2. Suppose f is differentiable on an interval I. Prove that f' is a pointwise limit of a sequence of continuous functions.

3. Prove Dini's theorem: if (f_n) is a monotone sequence of continuous functions on a closed bounded interval I that converges pointwise to a continuous function f, then $f_n \rightrightarrows f$.

4. Let $[a, b]$ be a (closed bounded) interval and let (c_n) be a sequence diverging to $+\infty$.
(a) Prove that $\int_a^b \sin(c_n x)\,dx \longrightarrow 0$ as $n \longrightarrow \infty$.
(b) Prove the Riemann-Lebesgue's lemma: For any continuous function f on a closed bounded interval $[a, b]$, $\int_a^b f(x)\sin(c_n x)\,dx \longrightarrow 0$ as $n \longrightarrow \infty$. (*Hint:* Approximate f by step functions.)

5. Prove that the series $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ converges uniformly on $\mathbb{R}$.

6. Find the set of x for which the series $\sum_{n=0}^{\infty} 2^n \sin^n x$ converges, and find the sum of this series on this set.

7. Prove that the zeta function $\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$, $x > 1$, is infinitely differentiable on $(1, +\infty)$.

8. Prove that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is an even function, then $a_n = 0$ for all odd n, and if f is an odd function, then $a_n = 0$ for all even n.

9. Find each of the following sums.
(i) $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$
(ii) $1 - x^3 + x^6 - x^9 + \ldots$, $|x| < 1$.
(iii) $\frac{x^2}{2} - \frac{x^3}{3\cdot 2} + \frac{x^4}{4\cdot 3} - \frac{x^5}{5\cdot 4} + \ldots$, $|x| < 1$.

10. Evaluate the following sums:
(i) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} \pi^{2n}}{(2n)!}$.
(ii) $\sum_{n=0}^{\infty} \frac{1}{(2n)!}$.
(iii) $\sum_{n=0}^{\infty} \frac{1}{(2n+1)2^n}$.
(iv) $\sum_{n=0}^{\infty} \frac{1}{3^n(n+1)}$.

11. If $f(x) = (\sin x)/x$ and $f(0) = 1$, find $f^{(k)}(0)$, $k \in \mathbb{N}$.

12. Let $\alpha \in \mathbb{R}$.
(a) Let $f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$, $|x| < 1$. Prove that $(1 + x)f'(x) = \alpha f(x)$.
(b) Prove that any function f satisfying the differential equation $(1+x)f'(x) = \alpha f(x)$ has form $f(x) = c(1+x)^\alpha$ for some $c \in \mathbb{R}$, and deduce "the binomial formula" $(1 + x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$, $|x| < 1$.

13. The Fibbonaci sequence is defined by $a_1 = a_2 = 1$ and $a_{n+2} = a_n + a_{n+1}$ for all $n \in \mathbb{N}$.
(a) Show that $a_{n+1}/a_n \leq 2$.
(b) Let $f(x) = \sum_{n=1}^{\infty} a_n x^{n-1}$. Prove that f is defined on $\left(\frac{-1}{2}, \frac{1}{2}\right)$.
(c) Prove that if $|x| < 1/2$, then $f(x) = \frac{1}{1-x-x^2}$.
(d) Decompose $\frac{1}{1-x-x^2}$ as $\frac{b_1}{c_1-x} + \frac{b_2}{c_2-x}$ to obtain another power series for f and prove that $a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$, $n \in \mathbb{N}$.

14. (a) Prove that the series $\sum 2^n \sin \frac{1}{3^n x}$ converges uniformly on $[a, +\infty)$ for any $a > 0$.

(b) By considering $\sum 2^n \sin \frac{1}{3^n x}$ for $x = \frac{2}{3^n \pi}$, show that the series doesn't converge uniformly on $(0, \infty)$.

(c) For $f(x) = \sum 2^n \sin \frac{1}{3^n x}$, $x > 0$, find (that is, express in the form of a series) f'.

15. Find $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)}$.

16. (a) Show that the series $\sum_{n=0}^{\infty} \left(\frac{x^{2n+1}}{2n+1} - \frac{x^{n+1}}{2n+2} \right)$ converges to $\frac{1}{2} \log(1 + x)$ locally uniformly on $(-1, 1)$, but converges to $\log 2$ at 1.

(b) Why doesn't this contradict Abel's theorem?

17. (a) Prove that for every $n \in \mathbb{N}$, $\int_0^{\pi} x \cos(nx)\, dx = \frac{-2}{n^2}$ if n is odd and 0 if n is even.

(b) Prove that for every $n \in \mathbb{N}$, $f_n(x) = 1 + 2\sum_{i=1}^{n} \cos(ix) = \sin\big((n + 1/2)x\big) / \sin(x/2)$. Prove that the function $x/\sin(x/2)$, $x \neq 0$, can be extended to 0 by continuity. Deduce that $\int_0^{\pi} x f_n(x)\, dx \longrightarrow 0$ as $n \longrightarrow \infty$. (*Hint:* Use the Riemann-Lebesgue lemma.)

(c) Combine (a) and (b) to prove that $\sum_{\text{odd } n \in \mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{8}$. Notice that $\sum_{\text{even } n \in \mathbb{N}} \frac{1}{n^2} = \frac{1}{4} \sum_{\text{all } n \in \mathbb{N}} \frac{1}{n^2}$ and deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.