Math 4181H **Midterm 2 review problems**

1. What can be said about the sequence (x_n) if it converges and $x_n \in \mathbb{Z}$ for all n?

2. Find $\lim \dfrac{n^n + 3n! + 10n^{10} + 2 \cdot 3^n}{5n! + 10^n - 2n^n + 4}$.

3. If (x_n) is a sequence with $x_n \geq 0$ for all n and $x_n \longrightarrow a$, prove that $\sqrt{x_n} \longrightarrow \sqrt{a}$.

4. (a) If the sequence (x_n) diverges to ∞ and the sequence (y_n) is bounded, prove that the sequence $(x_n + y_n)$ diverges to ∞.

(b) If the sequence (x_n) converges to 0 and the sequence (y_n) is bounded, prove that the sequence $(x_n y_n)$ converges to 0.

5. Let (x_n) be a sequence of real numbers. For each of the following statements, if it is true, say so; if false, give an example demonstrating this:

(i) If (x_n) converges it is bounded.

(ii) If (x_n) is bounded it converges.

(iii) If (x_n) is monotone it converges.

(iv) If (x_n) is monotone and bounded it converges.

(v) If (x_n) diverges to ∞ it is unbounded.

(vi) If (x_n) is unbounded it diverges to ∞.

(vii) If (x_n) is monotone and unbounded it diverges to ∞.

6. If the sequence (x_n) satisfies $1/n \leq x_n \leq n$ for all n, prove that $\lim \sqrt[n]{x_n} = 1$.

7. Prove that $\lim \sqrt[n]{n!} = +\infty$.

8. Let $a \in \mathbb{R}$, let $x_1 = a$ and $x_{n+1} = x_n^2 - x_n + 1$ for all $n \in \mathbb{N}$. Prove that the sequence (x_n) converges if $0 \leq a \leq 1$ and diverges to $+\infty$ otherwise.

9. (a) Let (x_n) and (y_n) be two sequences such that $|y_n - y_m| \leq |x_n - x_m|$ for all $n, m \in \mathbb{N}$. Prove that if (x_n) converges then (y_n) also converges.

(b) Give an example of two sequences (x_n) and (y_n) such that $|y_{n+1} - y_n| \leq |x_{n+1} - x_n|$ for all $n \in \mathbb{N}$, (x_n) converges but (y_n) diverges.

10. Consider the sequence $(x_n) = \left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \ldots\right)$. For which numbers α is there a subsequence of (x_n) converging to α? Also find $\limsup x_n$ and $\liminf x_n$.

11. For $d \in \mathbb{N}$, construct a sequence that has exactly d limit points.

12. (a) If $\lim x_n = a$ prove that every subsequence of (x_n) converges to a.

(b) Suppose a sequence (x_n) and $a \in \mathbb{R}$ are such that every subsequence of (x_n) has a subsequence that converges to a. Prove that $\lim x_n = a$.

13. Find $\lim\big(1 + 1/(2n)\big)^n$.

14. If the sequence (x_n) satisfies $\limsup |x_n| = 0$, prove that it converges to 0.

15. Let (x_n) be a sequence of positive numbers. Prove that $\limsup(x_{n+1}/x_n) \geq \limsup \sqrt[n]{x_n}$ and $\liminf(x_{n+1}/x_n) \leq \liminf \sqrt[n]{x_n}$. If $\lim(x_{n+1}/x_n)$ exists, prove that $\lim \sqrt[n]{x_n}$ also exists.

16. Let $f\colon A \longrightarrow \mathbb{R}$ be a function, let a be a limit point of A. Prove that $\lim_{x\to a} f(x) = b$ iff for every *monotone* sequence (x_n) in $A \setminus \{a\}$ with $x_n \longrightarrow a$ one has $f(x_n) \longrightarrow b$.

17. (a) (Glueing two continuous functions together:) Suppose that g and h are continuous at a and that $g(a) = h(a)$. Define $f(x) = \begin{cases} g(x), & x \geq a \\ h(x), & x \leq a. \end{cases}$ Prove that f is continuous at a.

18. A function $f\colon A \longrightarrow \mathbb{R}$ is said to be *Lipschitz* at a point $a \in A$ if there is $C > 0$ such that $|f(x) - f(a)| \leq C|x - a|$ for all $x \in A$ in a neighborhood of a. Prove that if f is Lipschitz at a then f is continuous at a.

19. (a) Prove that if f is continuous at a then so is $|f|$.

(b) Prove that if f and g are continuous at a then so are $\max\{f, g\}$ and $\min\{f, g\}$.

(c) Prove that every continuous f can be written $f = g - h$, where g and h are nonnegative and continuous.

20. Find $\lim 2^{(1+1/n)^n}$.

21. Suppose A_n, $n \in \mathbb{N}$, are pairwise disjoint subsets of $\mathbb{R}$ with no limit points. (That is, for any n and any $a \in \mathbb{R}$ there is $\delta > 0$ such that $(a - \delta, a + \delta) \cap A_n \setminus \{a\} = \emptyset$, and $A_n \cap A_m = \emptyset$ for any distinct n and m). Define f by $f(x) = \begin{cases} 1/n, & x \in A_n \\ 0, & x \notin A_n \text{ for all } n. \end{cases}$ Prove that $\lim_{x\to a} f(x) = 0$ for all $a \in [0, 1]$. Deduce that f is discontinuous at every point of $\bigcup_{n=1}^{\infty} A_n$ and continuous at all other points of $\mathbb{R}$.

22. Let f be a monotone function that takes all rational values (that is, $\mathrm{Rng}(f) \supseteq \mathbb{Q}$). Prove that f is continuous.

23. A function $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ is said to be *even* if $g(-x) = g(x)$ for all $x \in \mathbb{R}$ and *odd* if $g(-x) = -g(x)$ for all $x \in \mathbb{R}$. Prove that every function f continuous on $\mathbb{R}$ can be written as $f = E + O$, where E is a continuous even function and O is a continuous odd function.

24. Let $A \subseteq \mathbb{R}$, let f be a function $A \longrightarrow \mathbb{R}$, let $a = \inf A$. Define the function $\widetilde{f}\colon (a, +\infty) \longrightarrow \mathbb{R}$ by $\widetilde{f}(x) = \sup\{f(z) : z \in A,\ z \leq x\}$. Prove that $\widetilde{f}$ is an increasing function. If f is an increasing function, prove that $\widetilde{f}$ is an extension of f.

25. Find an integer n such that $f(x) = x^3 - x + 3$ has a root in $[n, n + 1]$.

26. Suppose that f is continuous on $[a, b]$ and that $f(x) \in \mathbb{Q}$ for all $x \in [a, b]$. What can be said about f? (*Hint:* Use the I.V.T.)

27. Suppose f is continuous on $[0, 1]$ and $\mathrm{Rng}(f) \subseteq [0, 1]$. Prove that $f(x_0) = x_0$ for some $x_0 \in [0, 1]$.

28. Let f be any polynomial function. Prove that there is $x_0 \in \mathbb{R}$ such that $|f(x_0)| \leq |f(x)|$ for all $x \in \mathbb{R}$.

29. (a) Suppose that f is continuous on an interval $[a, b]$ and let c be any number. Prove that there is a point on the graph of f which is closest to $(c, 0)$. (The graph of a function $f\colon A \longrightarrow \mathbb{R}$ is the set $\{(x, f(x)) : x \in A\}$ in the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. The distance between two points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ is defined as $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.)

(b) Show that this assertion is not necessarily true if $[a, b]$ is replaced by (a, b).

(c) Show that this assertion is true if $[a, b]$ is replaced by $\mathbb{R}$.

30. Suppose that f is continuous on an interval (a, b) and $\lim_{x\to a^+} f(x) = \lim_{x\to b^-} f(x)$, which may be finite or $\pm\infty$. Prove that f has a maximum on all of (a, b) or a minimum on all of (a, b).

31. Suppose that function f is continuous but not uniformly continuous on an interval $[a, b)$. Prove that $\lim_{x\to b^-} f(x)$ does not exist or is infinite.

32. If a function f is uniformly continuous on an interval $[a, b]$ and on the interval $[b, c]$, prove that f is uniformly continuous on $[a, c]$.

33. A function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is said to be periodic with period a, where a is a positive real number, if $f(x + a) = f(x)$ for all $x \in \mathbb{R}$.

(b) Prove that every continuous periodic function is bounded and attains its maximal and minimal values.

(a) Prove that every continuous periodic function is uniformly continuous.

34. If a function $f\colon \mathbb{R} \longrightarrow [0, +\infty)$ satisfies $f(x + y) = f(x)f(y)$ for all x and y and is continuous at 0 prove that f is continuous (at all points).