Math 4181H **Midterm 3 review problems**

1. Suppose that f is a polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ with critical points $-1, 1, 2, 3, 4$ and corresponding values $6, 1, 2, 4, 3$. Sketch the graph of f in the case n is even and in the case n is odd.

2. Let f be a differentiable function on $\mathbb{R}$. If f' is odd, prove that f is even; if f' is even and $f(0) = 0$, prove that f is odd.

3. Let $f(x) = x/2 + x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Prove that f is strictly increasing at 0 but not increasing in any neighborhood of 0.

4. If f is convex on an open interval I and differentiable at $a \in I$, prove that $f(x) \geq f(a) + f'(a)(x - a)$ for all $x \in I$.

5. Find $\tan'$ and $\arctan'$.

6. Find f' in terms of g and g' if

(i) $f(x) = g(x + g(a))$.

(ii) $f(x) = g(xg(a))$.

(iii) $f(x) = g(x + g(x))$.

(iv) $f(x) = g(xg(x))$.

7. Let $f(x) = (\sin x)/x$ for $x \neq 0$ and $f(0) = 1$. Find $f'(0)$ and $f''(0)$.

8. Prove that it is impossible to write $x = f(x)g(x)$ where f and g are differentiable at 0 and $f(0) = g(0) = 0$.

9. Let $f(x) = x^3 - 3x + a$ for some $a \in \mathbb{R}$. Prove that f cannot have more than one root in $[-1, 1]$.

10. If f is twice differentiable with $f(0) = 0$, $f(1) = 1$, and $f'(0) = f'(1) = 0$, then $|f''(x)| \geq 4$ for some $x \in (0, 1)$.

11. If f is invertible, differentiable, satisfies $f' = f^2$ on an interval I, and $f(x) \neq 0$ for all $x \in I$, find $(f^{-1})'$, f^{-1}, and f.

12. Prove that the function $f(x) = 1/x$ is convex on $(0, +\infty)$ and use this to prove that for any $n \in \mathbb{N}$ and $x_1, \ldots, x_n > 0$, $\frac{x_1+\cdots+x_n}{n} \geq \left(\frac{x_1^{-1}+\cdots+x_n^{-1}}{n}\right)^{-1}$.

13. Prove that of all rectangles with given perimeter, the square has the greatest area.

14. Suppose that $f\colon [0, 1] \longrightarrow [0, 1]$ is continuous on $[0, 1]$, differentiable on $(0, 1)$, and $f'(x) \neq 1$ for all $x \in [0, 1]$. Show that there is exactly one $x \in [0, 1]$ such that $f(x) = x$.

15. Suppose function f is continuous on $[0, +\infty)$, differentiable on $(0, +\infty)$, $f(0) = 0$, and f' is increasing on $(0, +\infty)$. Prove that the function $g(x) = f(x)/x$ is increasing on $(0, +\infty)$.

16. Let f be continuous on $[a, b]$ and differentiable on (a, b).

(a) Prove that if $f'(x) \geq M$ for all $x \in (a, b)$, then $f(b) \geq f(a) + M(b - a)$.

(b) Prove that if $f'(x) \leq M$ for all $x \in (a, b)$, then $f(b) \leq f(a) + M(b - a)$.

17. Suppose that the functions f and g are differentiable on an interval I, $f'(x) \geq g'(x)$ for all $x \in I$ and $f(a) = g(a)$ for some $a \in I$. Show that $f(x) \geq g(x)$ for all $x \in I$ with $x > a$ and $f(x) \leq g(x)$ for all $x \in I$ with $x < a$.

18. Suppose f is continuous on $[0, 1]$ and differentiable on $(0, 1)$ with $f'(x) \geq M > 0$ for all $x \in [0, 1]$. Show that there is an interval of length $\frac{1}{4}$ where $|f| \geq M/4$. (*Hint:* Consider two cases, $f(1/2) \geq 0$ and $f(1/2) \leq 0$.)

19. If a function f is differentiable on an interval I and has a unique critical point $a \in I$, prove that either f is monotone on I or attains the (global on I) maximum or minimum at a.

20. Let $c \neq 0$ and let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a differentible dunction satisfying $f'(x) = cf(x)$. Prove that $f(x) = ae^{cx}$ for some $a \in \mathbb{R}$.

21. Suppose that f is a differentiable function on $(0, +\infty)$ such that $f'(x) = 1/x$ for all $x > 0$ and $f(1) = 0$. Prove that $f(xy) = f(x) + f(y)$ for all $x, y > 0$ (and so, $f(x) = \log_a x$ for some $a > 0$). (*Hint:* Fix $y > 0$, consider the function $g(x) = f(xy)$, and find g'.)

22. Let f be a function differentiable on $(0, +\infty)$ such that $\lim_{x\to+\infty} f'(x) = c > 0$.

(a) Prove that $\lim_{x\to+\infty} f(x)/x = c$.

(b) Does this imply that $f(x) - cx$ is a bounded function?

23. (a) Give an example of a function f differentiable on $(0, +\infty)$ for which a finite $\lim_{x\to+\infty} f(x)$ exists, but $\lim_{x\to+\infty} f'(x)$ does not exist.

(b) Prove that if $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} f'(x)$ both exist and are finite, then $\lim_{x\to+\infty} f'(x) = 0$.

(c) Prove that if $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} f''(x)$ both exist and are finite, then $\lim_{x\to+\infty} f''(x) = 0$.

24. (a) Prove that for any $x, y \in \mathbb{R}$ such that $\cos x, \cos y, \cos(x + y) \neq 0$, $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$.

(b) arctan is the inverse of $\tan|_{(-\frac{\pi}{2},\frac{\pi}{2})}$. Prove that $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$, indicating all the necessary restrictions on x and y.

25. Prove that $\arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4}$.

26. If a function f is twice differentiable on $\mathbb{R}$ and satisfies $f'' = f$, prove that $f(x) = ae^x + be^{-x}$ for some $a, b \in \mathbb{R}$.

27. Find $(f^{-1})''(f(a))$ in terms of the derivatives of f at a.

28. Suppose that f is continuous on $[a, b]$, n-times differentiable on (a, b), and that $f(x) = 0$ for $n + 1$ distinct points x in $[a, b]$. Prove that $f^{(n)}(x) = 0$ for some x in (a, b).

29. Let $n \in \mathbb{N}$, and let $f(x) = x^n$ for $x > 0$ and $f(x) = 0$ for all $x \leq 0$. Prove that f is $(n - 1)$-times differentiable but not n-times differentiable at 0.

30. (a) Let $n \in \mathbb{N}$ and $f(x) = x^{2n} \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Prove that f is n times differentiable on $\mathbb{R}$ and that $f^{(n)}$ is discontinuous at 0.

(b) Let $n \in \mathbb{N}$ and $f(x) = x^{2n+1} \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Prove that f is n times differentiable on $\mathbb{R}$, and that $f^{(n)}$ is Lipschitz but not differentiable at 0.

31. If f and g are n times differentiable at a, prove by induction that $(fg)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) g^{(n-k)}(a)$ (Leibniz's formula).

32. Find $\displaystyle\lim_{x\to 0} \frac{\sin(x^2) - x^2}{(\cos x - 1)^3}$.

33. Given $c \in \mathbb{R}$, find $\displaystyle\lim_{x\to 0} (1 + cx)^{1/x}$.