Math 4181H **Midterm 4 review problems**

1. (a) If the function $f\colon[a,b]\longrightarrow\mathbb{R}$ has the property that $U(f,P)=L(f,P)$ for every partition P of $[a,b]$, what can be said about f?

(b) If the function $f\colon[a,b]\longrightarrow\mathbb{R}$ has the property that $U(f,P)=L(f,P)$ for some partition P of $[a,b]$, what can be said about f?

2. Prove that every monotone function on a closed bounded interval is integrable.

3. If $\int_a^b f=p$ and $\int_a^b g=q$, find $\int_a^b\big(\int_a^b f(x)g(y)\,dx\big)\,dy$.

4. Let f be the Riemann function ($f(x)=\frac{1}{n}$ if $x=\frac{m}{n}\in\mathbb{Q}$ in lowest terms and $f(x)=0$ if x is irrational). Prove that f is integrable on $[0,1]$ and $\int_0^1 f=0$.

5. Find integrable functions whose composition is not integrable.

6. Let f be bounded on $[a,b]$.

(a) For any partition P of $[a,b]$, prove that $\Delta(|f|,P)\le\Delta(f,P)$.

(b) If f is intergable, prove that so is $|f|$.

(c) If f and g are integrable on $[a,b]$, prove that so are $\max(f,g)$ and $\min(f,g)$.

(d) Prove that f is integrable iff $f^+=\max(f,0)$ and $f^-=\min(f,0)$ are integrable.

7. Suppose that f is locally integrable on $[0,+\infty)$ and $\lim_{x\to+\infty}f(x)=a$. Prove that $\lim_{x\to+\infty}\frac{1}{x}\int_0^x f(t)\,dt=a$.

8. If f is integrable on $[a,b]$, prove that $\int_a^b f(x)\,dx=\int_a^b f(b+a-x)\,dx$.

9. Find $\int_0^1 L(x)\,dx$ where L is the Cantor ladder function.

10. If $f\colon\mathbb{R}\longrightarrow\mathbb{R}$ is periodic with period a and integrable on $[0,a]$, show that for any $b\in\mathbb{R}$, f is integrable on $[b,b+a]$ and $\int_b^{b+a}f=\int_0^a f$.

11. Suppose that f is integrable on $[a,b]$. Prove that there is $c\in[a,b]$ such that $\int_a^c f=\int_c^b f$. Show that it is not always possible to choose c to be in (a,b).

12. Use the Fundamental Theorem of Caluclus and Darboux's theorem to give another proof of the Intermediate Value Theorem.

13. Let f be a continuous function on $[a,b]$ with the property that $\int_a^b fg=0$ for all continuous functions g on $[a,b]$ satisfying $g(a)=g(b)=0$. Prove that $f=0$.

14. Let the function f be integrable and ≥ 0 on an interval $[a,b]$ and suppose there is $c\in[a,b]$ such that f is continuous at c and $f(c)>0$. Prove that $\int_a^b f>0$.

15. Let f be integrable on $[a,b]$, let $c\in(a,b)$, and let $F(x)=\int_a^x f\,dt$, $x\in[a,b]$. Prove or disprove:

(i) If f is differentiable at c, then F is differentiable at c.

(ii) If f is differentiable at c, then F' is continuous at c.

(iii) If f' is continuous at c, then F' is continuous at c.

16. Find $(f^{-1})'(0)$ if $f(x)=\int_0^x(1+\sin(\sin t))\,dt$.

17. (a) Find the derivatives of $F(x)=\int_1^x\frac{1}{t}dt$ and $G(x)=\int_b^{bx}\frac{1}{t}dt$.

(b) Use (a) to prove that for $a>1$ and $b>0$, $\int_1^a 1/t\,dt=\int_b^{ab}1/t\,dt$.

(c) For $a,b>1$ prove that $\int_1^a\frac{dt}{t}+\int_1^b\frac{dt}{t}=\int_1^{ab}\frac{dt}{t}$.

18. Let $f(x)=\sin(1/x)$, $x\ne 0$, and $f(0)=0$. Is the function $F(x)=\int_0^x f$ differentiable at 0? (*Hint:* Differentiate the function $G(x)=x^2\cos(1/x)$ and compare G' and f.)

19. Prove that for every integer $n\ge 2$, $\int_0^{\pi/2}\sin^n x\,dx=\frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx$.

20. Compute the improper integral $\int_0^1 \log x \, dx$.

21. Prove the following version of the integration by parts for improper integrals: If f and g are locally integrable on $[a, \beta)$ (where $\beta \in \mathbb{R} \cup \{+\infty\}$) and have primitives $F = \int f$ and $G = \int g$ on $[a, \beta)$, $\lim_{x \to \beta^-} F(x)G(x)$ exists, Fg has a ptimitive, and $\int_a^\beta F(x)g(x)\, dx$ converges, then $\int_a^\beta f(x)G(x)\, dx = \lim_{x\to\beta^-} F(x)G(x) - F(a)G(a) - \int_a^\beta F(x)g(x)\, dx$.

22. Determine if the improper integral $\int_0^\infty \sin(x^2)\, dx$ converges.

23. Using the fact that $\int_0^{+\infty} e^{-x^2} dx = \sqrt{\pi}/2$, find $(-1/2)! = \Gamma(1/2)$ and $(1/2)! = \Gamma(3/2)$.

24. Find the Taylor polynomial $P_{a,n,f}$ for

(a) $f(x) = e^{\sin x}$, $a = 0$, $n = 3$.

(b) $f(x) = e^x$, $a = 1$.

(c) $f(x) = x^5 + x^3 + x$, $a = 0$, $n = 4$.

25. (a) For every $n \in \mathbb{N}$, find the Taylor polynomial $P_{0,4n+2,f}$ for $f(x) = \sin(x^2)$.

(b) Find $f^{(k)}(0)$ for all k.

(c) In general, if $f(x) = g(x^m)$, find $f^{(k)}(0)$ in terms of the derivatives of g at 0.

26. Find $P_{0,5,f}$ for

(a) $f(x) = e^x \sin x$.

(b) $f(x) = \tan x$.

27. Find (a) $\lim_{x\to 0} \frac{e^x - 1 - x - \frac{1}{2}x^2}{x - \sin x}$.

(b) $\lim_{x\to 0}\left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right)$.

(c) $\lim_{x\to 0}\left(\frac{1}{\sin^2 x} - \frac{1}{\sin(x^2)}\right)$.

28. Use Taylor polynomials to reprove Leibniz's identity $(fg)^{(n)}(a) = \sum_{i=0}^n \binom{n}{i} f^{(n-i)}(a) g^{(i)}(a)$.

29. Let $f(x) = \begin{cases} \frac{e^x - 1}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$ Taking it for granted that f is infnitely differentiable on $\mathbb{R}$, find the Taylor polynomial of degree n for f at 0, and compute $f^{(n)}(0)$ for all n.

30. Define $f(x) = \frac{\log(1+x^2)}{x^2}$ for $x \neq 0$ and $f(0) = 1$. Taking it for granted that f is infnitely differentiable on $\mathbb{R}$, find the derivative $f^{(100)}(0)$.

31. Using the fact that $\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3}$, show that $\pi = 3.14159...$