Math 4181H **Solutions to Homework 1**

Chapter 1, pp. 13-18:

1. *Prove the following:*

5pt (ii) *For any $x, y \in \mathbb{R}$, $x^2 - y^2 = (x-y)(x+y)$.*

Solution. Applying (P9), (P8), (P1), and the definition of $-y$, we get

$$(x-y)(x+y) = (x+(-y))(x+y) = (x+(-y))x + (x+(-y))y = x^2 + (-y)x + xy + y(-y).$$

Now, using the fact that $(-y)x = -xy$ (proven in lecture notes), we get

$$x^2 + (-y)x + xy + y(-y) = x^2 - xy + xy - y^2 = x^2 - y^2.$$

5pt (iii) *If $x^2 = y^2$, then $x = y$ or $x = -y$.*

Solution. By (ii), we have $x^2 - y^2 = (x-y)(x+y)$, so if $x^2 - y^2 = 0$, then $(x-y)(x+y) = 0$, so $x - y = 0$ or $x + y = 0$ (as it was proven in lecture notes). In the first case, $x + (-y) = 0$, so $x = -(-y) = y$ (proven in lecture notes); in the second case, $x = -y$.

Cf. 2. *What is wrong with the following proofs?*

5pt (a) *Let $x = y \neq 0$. Then $x^2 = xy$, so $x^2 - y^2 = xy - y^2$, so $(x+y)(x-y) = y(x-y)$, so $x + y = y$, so $2y = y$, so $2 = 1$.*

Solution. The mistake in the proof is that $(x+y)(x-y) = y(x-y)$ does not imply that $x + y = y$ (since $x - y = 0$).

5pt (b) *If $2 = 1$ then $2 \cdot 2 = 2 \cdot 1$ then $4 = 2$ then $4 - 3 = 2 - 3$ then $1 = -1$ then $1^2 = (-1)^2$ then $1 = 1$, which is true. So, $2 = 1$.*

Solution. If a statement P implies a true statement, it doesn't mean that P is true itself.

In contrast, if a true statement implies P, then P is true. (If you can derive from $1 = 1$ that $2 = 1$, then $2 = 1$ indeed. But you cannot.)

We define a/b (as well as $a : b$ and $\frac{a}{b}$) as ab^{-1}.

3. *Let $a, b, c, d \in \mathbb{R}$. Prove the following:*

3pt (i) *If $b, c \neq 0$, then $a/b = (ac)/(bc)$.*

Solution. Repeatedly applying axioms (P5) and (P6) (and implicitly (P8)),

$$\begin{aligned}(ac)/(bc) = (ac)(bc)^{-1} = (ac)(b^{-1}c^{-1}) = a(c(b^{-1}c^{-1})) = a((cb^{-1})c^{-1}) = a((b^{-1}c)c^{-1}) = a(b^{-1}(cc^{-1}))\\ = a(b^{-1}1) = ab^{-1} = a/b.\end{aligned}$$

3pt (ii) *if $b, d \neq 0$, then $a/b + c/d = (ad + bc)/(bd)$.*

Solution. By (i), $\frac{a}{b} = \frac{ad}{bd}$ and $\frac{c}{d} = \frac{bc}{bd}$. So,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = (ad)(bd)^{-1} + (bc)(bd)^{-1} = (ad+bc)(bd)^{-1} = \frac{ad+bc}{bd}.$$

3pt (vi) *If $b, d \neq 0$, then $a/b = c/d$ iff $ad = bc$.*

Solution. By (i), $\frac{a}{b} = \frac{ad}{bd}$ and $\frac{c}{d} = \frac{bc}{bd}$. So, $\frac{a}{c} = \frac{b}{d}$ iff $\frac{ad}{cd} = \frac{bc}{cd}$ iff $(ad)(cd)^{-1} = (bc)(cd)^{-1}$ iff $ad = bc$ by the cancellation property.

3pt (vii) *If $a, b \neq 0$, prove that $a/b = b/a$ iff $a = b$ or $a = -b$.*

Solution. By (vi), $\frac{a}{b} = \frac{b}{a}$ iff $a^2 = b^2$, which, by (1)(iii), holds iff $a = b$ or $a = -b$.

For $a, b \in \mathbb{R}$ we define $a < b$ if $b - a \in P$ (that is, is positive), $a > b$ if $b < a$, $a \le b$ if $a < b$ or $a = b$, and $a \ge b$ if $b \le a$.

5. *Let $a, b, c, d \in \mathbb{R}$. Prove the following:*

2pt (i) *If $a < b$ and $c < d$ then $a + c < b + d$.*

Solution. If $a < b$ then $a + c < b + c$ (proven in lecture notes), and, similarly, if $c < d$ then $b + c < b + d$. So, $a + c < b + d$.

2pt (iii) *If $a < b$ and $c > d$ then $a - c < b - d$.*

Solution. If $c > d$ then $-c < -d$ (proven in lecture notes), so by (i), $a - c = a + (-c) < b + (-d) = b - d$.

2pt (v) *If $a < b$ and $c < 0$ then $ac > bc$.*

Solution. If $c < 0$, then $-c > 0$. We proved that in this case $a(-c) < b(-c)$, so $-ac < -bc$, so $ac > bc$.

2pt (vi) *If $a > 1$ then $a^2 > a$.*

Solution. Since $a > 1$, we have $a > 0$, so, since $a > 1$, we get $a^2 = aa > 1a = a$.

2pt (vii) *If $0 < a < 1$ then $a^2 < a$.*

Solution. Since $a > 0$, and $a < 1$, we get $a^2 = aa < 1a = a$.

2pt (viii) *If $0 \le a < b$ and $0 \le c < d$, then $ac < bd$.*

Solution. Since $a < b$ and $c > 0$, we have $ac < bc$; since $c < d$ and $b > 0$, we have $bc < bd$. So, $ac < bd$.

2pt (ix) *If $0 \le a < b$ then $a^2 < b^2$.*

Solution. This is a special case of (viii), with $c = a$ and $d = b$.

2pt (x) *If $a, b \ge 0$ and $a^2 < b^2$, then $a < b$.*

Solution. We can deduce this from (ix) "by contraposition": assume that $a < b$ is not true, and show that $a^2 < b^2$ is not true. Indeed, if $a = b$ then $a^2 = b^2$, and if $a > b$ then $a^2 > b^2$ by (ix).

10pt **8.** *Let F be a field (that is, a set with two operations satisfying (P1)–(P9)) with an order "$<$" satisfying (P10′)–(P12′) (from my Lecture Notes) or (P′10)-(P′13) from exercise 8 in Spivak. Define $P = \{a \in F : a > 0\}$. Prove that P satisfies (P10)–(P12).*

Solution. Assume
(P10′) for any two numbers $a, b \in \mathbb{R}$ exactly one of the following holds: either $a < b$, or $a = b$, or $b < a$;
(P10″) if $a < b$ and $b < c$ then $a < c$;
(P11′) if $a < b$ then for any c, $a + c < b + c$;
(P12′) if $a < b$ then for any $c > 0$, $ac < bc$.

Let $P = \{a \in \mathbb{R} : a > 0\}$.

For any $a \in \mathbb{R}$, taking $b = 0$ in (P10′), we see that exactly one of the following is true: either $a < 0$, or $a = 0$, or $0 < a$. We have $0 < a$ iff $a \in P$. If $a < 0$, then adding $-a$ to both parts and using (P11′) we get $0 = a - a < 0 - a = -a$, so $-a \in P$; conversely, if $-a \in P$, then $0 < -a$, and adding a to both parts we get $a < 0$. So, exactly one of the following is true: either $a \in P$, or $a = 0$, or $-a \in P$, which proves (P10).

Let $a, b \in P$. Then $a > 0$, so by (P11′) $a + b > 0 + b = b$, and since also $b > 0$, by (P10″) we have $a + b > 0$, so $a + b \in P$. This proves (P11).

Let $a, b \in P$. Then $a > 0$ and $b > 0$, so by (P12′), $ab > 0b = 0$, so $ab \in P$. This proves (P12).

12. *Recall that, for $x \in \mathbb{R}$, the absolute value $|x|$ is defined by $|x| = x$ if $x \ge 0$ and $|x| = -x$ if $x < 0$. Prove the following:*

2pt (ii) *For any $x \in \mathbb{R} \setminus \{0\}$, $|1/x| = 1/|x|$.*

Solution. If $x > 0$, then $1/x > 0$. (Proven in lecture notes.) So, $|1/x| = 1/x = 1/|x|$.

If $x < 0$, then $1/x < 0$. So, $|1/x| = -(1/x) = 1/(-x) = 1/|x|$.

2pt (iii) *For any $x \in \mathbb{R}$ and $y \in \mathbb{R} \setminus \{0\}$, $|x|/|y| = |x/y|$.*

Solution. $|x/y| = |x| \cdot |y^{-1}| = |x| \cdot |y|^{-1} = |x|/|y|$.

2pt (iv) *For any $x, y \in \mathbb{R}$, $|x - y| \le |x| + |y|$.*

Solution. $|x - y| = |x + (-y)| \le |x| + |-y| = |x| + |y|$.

2pt (v) *For any $x, y \in \mathbb{R}$, $|x| - |y| \le |x - y|$.*

Solution. $|x - y| + |y| \ge |x - y + y| = |x|$, so $|x| - |y| \le |x - y|$.

2pt (vi) *For any* $x, y \in \mathbb{R}$, $\big||x| - |y|\big| \le |x - y|$.

Solution. We have both $|x| - |y| \le |x - y|$ and $-(|x| - |y|) = |y| - |x| \le |y - x| = |x - y|$, so $|x| - |y| \ge -|x - y|$, so $\big||x| - |y|\big| \le |x - y|$.

5pt **20.** *Prove that if* $x, x_0, y, y_0, \varepsilon$ *are real numbers such that* $|x - x_0| < \varepsilon/2$ *and* $|y - y_0| < \varepsilon/2$, *then* $|(x + y) - (x_0 + y_0)| < \varepsilon$ *and* $|(x - y) - (x_0 - y_0)| < \varepsilon$.

Solution.

$$\big|(x+y) - (x_0 + y_0)\big| = \big|(x - x_0) + (y - y_0)\big| \le |x - x_0| + |y - y_0| < \varepsilon/2 + \varepsilon/2 = \varepsilon/2(1+1) = \varepsilon \cdot 2^{-1} \cdot 2 = \varepsilon \cdot 1 = \varepsilon.$$

$$\big|(x - y) - (x_0 - y_0)\big| = \big|(x - x_0) - (y - y_0)\big| \le |x - x_0| + |y - y_0| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$