Math 4181H **Solutions to Homework 10**

15pt **A1.** *For each of the following series determine whether it converges absolutely, converges conditionally, diverges to ∞, or just diverges (and name the test that proves your claim):*
(a) $\sum \frac{(-1)^n}{n^{1/n}}$; (b) $\sum \frac{(-1)^n n^2}{2^n}$; (c) $\sum \frac{(-1)^n}{\sqrt{n}}$; (d) $\sum \frac{\sin n}{\log n}$; (e) $\sum \sin(1/n)$.

Solution.
(a) $\sum \frac{(-1)^n}{n^{1/n}}$ diverges by the vanishing test, since $\left|\frac{(-1)^n}{n^{1/n}}\right| = \frac{1}{|n^{1/n}|} \longrightarrow 1 \neq 0$.

(b) $\sum \frac{(-1)^n n^2}{2^n}$ converges absolutely by the root test, since $\sqrt[n]{\left|\frac{(-1)^n n^2}{2^n}\right|} = \frac{\sqrt[n]{n}^2}{2} \longrightarrow \frac{1}{2} < 1$.

(c) $\sum \frac{(-1)^n}{\sqrt{n}}$ diverges absolutely by comparison test, since $\left|\frac{(-1)^n}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}} > \frac{1}{n}$ and $\sum \frac{1}{n} = \infty$, and converges conditionally by Leibniz's test, since $\frac{1}{\sqrt{n}} \searrow 0$.

(d) $\sum \frac{\sin n}{\log n}$ converges conditionally by Dirichlet's test, since $\frac{1}{\log n} \searrow 0$ and the sequence of partial sums of $\sin n$ are bounded.

To show that this series doesn't converge absolutely, let $\varepsilon = \sin\frac{\pi}{10}$; then $|\sin x| < \varepsilon$ iff $|x - k\pi| < \frac{\pi}{10}$ for some $k \in \mathbb{Z}$. If $n \in \mathbb{N}$ is such that $k\pi - \frac{\pi}{10} < n < k\pi + \frac{\pi}{10}$ for some $k \in \mathbb{Z}$, then $k\pi + \frac{\pi}{10} < n+1 < (k+1)\pi - \frac{\pi}{10}$, so, if $|\sin n| < \varepsilon$, then $|\sin(n+1)| > \varepsilon$. It follows that for any n, $\frac{|\sin n|}{\log n} + \frac{|\sin(n+1)|}{\log(n+1)} > \frac{|\sin n|}{\log(n+1)} + \frac{|\sin(n+1)|}{\log(n+1)} > \frac{\varepsilon}{\log(n+1)}$, and so

$$\sum_{n=2}^{\infty}\left|\frac{\sin n}{\log n}\right| = \sum_{n=1}^{\infty}\Big(\frac{|\sin 2n|}{\log 2n} + \frac{|\sin(2n+1)|}{\log(2n+1)}\Big) \geq \sum_{n=1}^{\infty} \frac{\varepsilon}{\log(2n+1)} = \infty$$

(since $\frac{1}{\log(2n+1)} > \frac{1}{n}$ for all n large enough).

(e) $\sin(1/n)/(1/n) \longrightarrow 1$, $\sum \frac{1}{n} = \infty$, so $\sum \sin(1/n) = \infty$ by the limit comparison test.

5pt **A2.** *Prove that the series $\sum \frac{1}{n(\log n)(\log(\log n))^\alpha}$ converges for $\alpha > 1$ and diverges for $0 < \alpha \leq 1$.*

Solution. I'll use the integral test (the condensation test also works). The function $f(x) = \frac{1}{x(\log x)(\log(\log x))^\alpha}$ is positive and decreasing on $[100, +\infty)$, with $f(n) = \frac{1}{n(\log n)(\log(\log n))^\alpha}$ for all $n \geq 100$, so the series converges iff the improper integral $\int_{100}^{\infty} f$ does. (Why 100? To be sure that $\log(\log x)$ is defined.) We have

$$\int_{100}^{\infty} \frac{dx}{x(\log x)(\log(\log x))^\alpha} = \int_{100}^{\infty} \frac{d\log x}{\log x(\log(\log x))^\alpha} = \int_{100}^{\infty} \frac{d\log x}{\log x(\log(\log x))^\alpha} = \int_{100}^{\infty} \frac{d\log(\log x)}{(\log(\log x))^\alpha} = \int_{\log\log 100}^{\infty} \frac{dy}{y^\alpha}\Big|_{y=\log\log x},$$

which converges for $\alpha > 1$ and diverges for $\alpha \leq 1$.

10pt **A3.** *If Λ is an uncountable set, $\Lambda \longrightarrow \mathbb{R}$, $i \mapsto a_i$, is a mapping, and $a_i > 0$ for all i, prove that $\sum a_i = \infty$. (That is, prove that for any $M \in \mathbb{R}$ there exists a finite set $F \subset \Lambda$ such that $\sum_{i \in F} a_i > M$.)*

Solution. For every $n \in \mathbb{N}$ let $P_n = \{i \in \Lambda : a_i > 1/n\}$. Since for every i there is n such that $a_i > 1/n$, so $a_i \in P_n$, we have $\bigcup_{n=1}^{\infty} P_n = \Lambda$. If all P_n are finite, then Λ is at most countable. Hence, for some n, P_n is infinite (and even uncountable). For any $N \in \mathbb{N}$ choose nN elements $a_1, \ldots, a_{nN} \in P_n$, then $a_1 + \cdots + a_{nN} > nN\frac{1}{n} = N$.

Chapter 23, pp. 489-498:

10pt **10.** *If $a_i = b_i = \frac{(-1)^i}{\sqrt{i}}$, $i \in \mathbb{N}$, prove that the series $\sum a_i$ (and $\sum b_i$) converge but their Cauchy product $\sum_{k=2}^{\infty}\big(\sum_{i=1}^{k-1} a_i b_{k-i}\big)$ diverges.*

Solution. The series $\sum \frac{(-1)^i}{\sqrt{i}}$ converges (conditionally) by the alternating test.

For any $0 < i < k$, $\sqrt{i(k-i)} \leq (i + (k-i))/2 = k/2$; so, for any integer $k \geq 2$,

$$\Big|\sum_{i=1}^{k-1} \frac{(-1)^i}{\sqrt{i}} \cdot \frac{(-1)^{k-i}}{\sqrt{k-i}}\Big| = \Big|(-1)^k \sum_{i=1}^{k-1} \frac{1}{\sqrt{i} \cdot \sqrt{k-i}}\Big| = \sum_{i=1}^{k-1} \frac{1}{\sqrt{i} \cdot \sqrt{k-i}} \leq (k-1)\frac{1}{k/2},$$

which $\not\longrightarrow 0$ as $k \longrightarrow \infty$. Hence, the series $\sum_{k=2}^{\infty}\big(\sum_{i=1}^{k-1} \frac{(-1)^i}{\sqrt{i}} \cdot \frac{(-1)^{k-i}}{\sqrt{k-i}}\big)$ diverges by the vanishing test.

10pt **A4.** *A function h on an interval $[a,b]$ is said to be a step function if there is a partition $a = x_0 < x_1 < \ldots < x_n = b$ of $[a,b]$ such that for every i, h is constant on the interval (x_{i-1}, x_i). Prove that for any closed bounded interval $[a,b]$ the step functions are dense in $C([a,b])$ with respect to the uniform norm: for every continuous function f on $[a,b]$ and any $\varepsilon > 0$ there exists a step function h on $[a,b]$ such that $\|f - h\| < \varepsilon$.*

Solution. Let $\varepsilon > 0$. Since $[a,b]$ is a closed bounded interval, f is uniformly continuous; let $\delta > 0$ be such that $|f(x) - f(y)| < \varepsilon$ when $|x - y| < \delta$. Let $P = \{x_0, \ldots, x_n\}$ be any partition of $[a,b]$ with $\operatorname{mesh} P < \delta$. Define $h(x) = f(x_i)$ for $x \in (x_{i-1}, x_i]$, $i = 1, \ldots, n$, and $h(x_0) = f(x_0)$. Then h is a step function, and for every $x \in [a,b]$, if i is sucj that $x \in (x_{i-1}, x_i]$, we have $|f(x) - h(x)| = |f(x) - f(x_i)| < \varepsilon$ since $|x - x_i| < \delta$.

A5. *Let (f_n) be a sequence of functions $A \longrightarrow \mathbb{R}$ and let $f_n \rightrightarrows f$.*

5pt (a) *If all f_n are uniformly continuous on A prove that f is uniformly continuous on A.*

Solution. Let $\varepsilon > 0$. Find n such that $\|f - f_n\| < \varepsilon/3$. Find $\delta > 0$ such that for any $x, y \in A$ with $|x - y| < \delta$, $|f_n(x) - f_n(y)| < \varepsilon/3$. Then for any $x, y \in A$ with $|x - y| < \delta$,

$$|f(x) - f(y)| \leq ||f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \varepsilon.$$

5pt (b) *If the sequence (f_n) is "uniformly Lipschitz on A", that is, if there is C such that $|f_n(x) - f_n(y)| \leq C|x - y|$ for all $n \in \mathbb{N}$ and all $x, y \in A$, prove that f is Lipschitz on A.*

Solution. Let $x, y \in A$. Let $\varepsilon > 0$; find n such that $\|f(x) - f_n(x)\| < \varepsilon/2$ and $\|f(y) - f_n(y)\| < \varepsilon/2$. Then

$$|f(x) - f(y)| \leq ||f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \varepsilon/2 + C|x - y| + \varepsilon/2 = C|x - y| + \varepsilon.$$

Since this is true for any $\varepsilon > 0$, $|f(x) - f(y)| \leq C|x - y|$.

Another solution. For any $x, y \in A$, since $|f_n(x) - f_n(y)| \leq C|x - y|$ for all n,

$$|f(x) - f(y)| = \lim_{n \to \infty} |f_n(x) - f_n(y)| \leq C|x - y|.$$

Chapter 24, pp. 517-525:

2. *Find the pointwise limit of (f_n) and decide whether (f_n) converges uniformly to this limit.*

5pt (i) $f_n(x) = x^n - x^{2n}$ *on* $[0,1]$.

Solution. For any $x \in [0,1]$, $x^n - x^{2n} \longrightarrow 0$ as $n \longrightarrow \infty$, so $f_n \longrightarrow 0$ on $[0,1]$. For any n, $f_n'(x) = nx^{n-1} - (2n)x^{2n-1}$, which is $= 0$ iff $x = \frac{1}{\sqrt[n]{2}}$. Since $f_n(0) = f_n(1) = 0$, $|f_n|$ attains its maximum at this point, so $\|f_n\|_{[0,1]} = \sup_{x \in [0,1]} |f_n(x)| = \big(\frac{1}{\sqrt[n]{2}}\big)^n - \big(\frac{1}{\sqrt[n]{2}}\big)^{2n} = 1/4$. (Or: $\max_{[0,1]} |x^{2n} - x^n| = \max_{[0,1]} |y^2 - y| = 1/4$.) Since $\|f_n\| \not\longrightarrow 0$, the convergence $f_n \longrightarrow 0$ is not uniform.

5pt (ii) $f_n(x) = \frac{nx}{1+n+x}$ *on* $[0, \infty)$.

Solution. For each $x \geq 0$, $\lim_{n \to \infty} \frac{nx}{1+n+x} = x$, so $f_n \longrightarrow f$ where $f(x) = x$. For any n,

$$|f_n(x) - f(x)| = \left| \frac{nx}{1+n+x} - x \right| = \frac{x + x^2}{1+n+x},$$

which is an unbounded function, so $\|f_n - f\| = \infty$. So, the convergence is not uniform.

5pt (iv) $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$ *on* $\mathbb{R}$.

Solution. For every x, $\lim_{n \to \infty} \sqrt{x^2 + \frac{1}{n^2}} = |x|$, so $f_n \longrightarrow f$ where $f(x) = |x|$. For any n, consider the function $g_n(x) = f_n(x) - f(x) = \sqrt{x^2 + \frac{1}{n^2}} - |x|$. This function is even, nonnegative, and for $x > 0$, $g_n'(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n^2}}} - 1 < 0$, so g_n is decreasing on $[0, \infty)$. Hence, $\|g_n\| = \sup_{x \in \mathbb{R}} |g_n(x)| = g_n(0) = \frac{1}{n}$, which tends to 0 as $n \longrightarrow \infty$. Thus, $f_n \rightrightarrows f$.

10pt **29.** (a) *Suppose that* (f_n) *is a sequence of continuous functions on* $[a, b]$ *that converges uniformly on* $[a, b]$ *to a function* f*, and let* (x_n) *be a sequence in* $[a, b]$ *with* $\lim x_n = c$*. Prove that* $\lim f_n(x_n) = f(c)$*.*

Solution. Since f_n are continuous and $f_n \rightrightarrows f$, f is also continuous. Let $\varepsilon > 0$. Find $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$. Find N such that for any $n \geq N$, $\|f_n - f\| < \varepsilon/2$ and $|x_n - c| < \delta$. Then for any $n \geq N$,

$$\big|f_n(x_n) - f(c)\big| \leq \big|f_n(x_n) - f(x_n)\big| + \big|f(x_n) - f(c)\big| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

5pt (b) *Is this statement true without assuming that* f_n *are continuous?*

Solution. Of course, not. Even if all f_n are equal to f, if f is discontinuous at c, then there is a sequence (x_n) such that $x_n \longrightarrow c$ but $f(x_n) \not\longrightarrow f(c)$.