Math 4181H **Solutions to Homework 2**

15pt **A1.** *Let $a, b \in \mathbb{R}$, $0 < a < b$; let $A = \frac{a+b}{2}$ (the airthmetic mean of a and b), $G = \sqrt{ab}$ (the geometric mean), $H = \left(\frac{a^{-1}+b^{-1}}{2}\right)^{-1}$ (the harmonic mean), and $Q = \sqrt{\frac{a^2+b^2}{2}}$ (the quadratic mean). Prove that $a < H < G < A < Q < b$.*

Solution. It is important that all the numbers appearing in the proof are positive.

We have $b^{-1} < a^{-1}$, so $\frac{a^{-1}+b^{-1}}{2} < \frac{2a^{-1}}{2} = a^{-1}$, so $H = \left(\frac{a^{-1}+b^{-1}}{2}\right)^{-1} > a$.

$G < A$ was proved in class: $A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}(a + b - 2\sqrt{ab}) = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 > 0$.

It follows that $\frac{a^{-1}+b^{-1}}{2} > \sqrt{a^{-1}b^{-1}} = \left(\sqrt{ab}\right)^{-1}$, so $H = \left(\frac{a^{-1}+b^{-1}}{2}\right)^{-1} < \sqrt{ab} = G$.

Next, $Q^2 - A^2 = \frac{a^2+b^2}{2} - \left(\frac{a+b}{2}\right)^2 = \frac{a^2+b^2}{2} - \frac{a^2+b^2+2ab}{4} = \frac{2a^2+2b^2}{4} - \frac{a^2+b^2+2ab}{4} = \frac{a^2+b^2-2ab}{4} = \frac{(a-b)^2}{4} > 0$, so $Q^2 > A^2$, so $Q > A$.

And finally, $Q = \sqrt{\frac{a^2+b^2}{2}} < \sqrt{\frac{b^2+b^2}{2}} = \sqrt{b^2} = b$.

5pt **A2.** (a) *Prove that for any $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{R}$, $\left|\sum_{i=1}^n a_i\right| \le \sum_{i=1}^n |a_i|$.*

Solution. For $n = 1$, the statement is true: $\left|\sum_{i=1}^1 a_i\right| = |a_1| = \sum_{i=1}^1 |a_i|$. Assume that it is true for some $n \in \mathbb{N}$, that is, $\left|\sum_{i=1}^n a_i\right| \le \sum_{i=1}^n |a_i|$ for any $a_1, \ldots, a_n \in \mathbb{R}$. Then for any $a_1, \ldots, a_{n+1} \in \mathbb{R}$ we have

$$\left|\sum_{i=1}^{n+1} a_i\right| = \left|\sum_{i=1}^n a_i + a_{n+1}\right| \underset{\text{(by triangle inequality)}}{\le} \left|\sum_{i=1}^n a_i\right| + |a_{n+1}| \underset{\text{(by induction hypothesis)}}{\le} \sum_{i=1}^n |a_i| + |a_{n+1}| = \sum_{i=1}^{n+1} |a_i|.$$

Hence, by induction, the statement is true for all n.

10pt (b) *Prove the following version of the Cauchy-Schwarz inequality: for any $n \in \mathbb{N}$ and $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$, $\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) \ge \left(\sum_{i=1}^n a_ib_i\right)^2$, with "=" only if $a_1 = \cdots = a_n = 0$ or $b_1 = \cdots = b_n = 0$ or there is $x \in \mathbb{R}$ such that $b_1 = a_1x$, …, $b_n = a_nx$.*

Solution. If $a_1 = \cdots = a_n = 0$ or $b_1 = \cdots = b_n = 0$, then $\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) = \left(\sum_{i=1}^n a_ib_i\right)^2 = 0$. If there is x such that $b_1 = a_1x$, …, $b_n = a_nx$, then

$$\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) = \left(\sum_{i=1}^n a_i^2\right)x^2\left(\sum_{i=1}^n a_i^2\right) = \left(\sum_{i=1}^n a_i^2\right)^2x^2 = \left(\sum_{i=1}^n a_i^2x\right)^2 = \left(\sum_{i=1}^n a_ixa_i\right)^2 = \left(\sum_{i=1}^n a_ib_i\right)^2.$$

Assume that not all of $a_i = 0$, then $\sum_{i=1}^2 a_i^2 > 0$. Consider the quadratic expression

$$\sum_{i=1}^n (a_ix - b_i)^2 = \left(\sum_{i=1}^2 a_i^2\right)x^2 - 2\left(\sum_{i=1}^2 a_ib_i\right)x + \left(\sum_{i=1}^2 b_i^2\right).$$

For every $x \in \mathbb{R}$ this expression is ≥ 0, and $= 0$ for some x only if $a_ix - b_i = 0$ for all i, that is, when $b_i = a_ix$ for all i. Hence, if such an x doesn't exist, the expression is always > 0. This implies that its discriminant, which is $4\left(\sum_{i=1}^2 a_ib_i\right)^2 - 4\left(\sum_{i=1}^2 a_i^2\right)\left(\sum_{i=1}^2 b_i^2\right)$, is < 0, and hence, $\left(\sum_{i=1}^2 a_ib_i\right)^2 < \left(\sum_{i=1}^2 a_i^2\right)\left(\sum_{i=1}^2 b_i^2\right)$.

5pt (c) *Prove the n-dimensional triangle inequality: for any $n \in \mathbb{N}$ and $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$, $\sqrt{\sum_{i=1}^n (a_i + b_i)^2} \le \sqrt{\sum_{i=1}^n a_i^2} + \sqrt{\sum_{i=1}^n b_i^2}$ with "=" only if $a_1 = \cdots = a_n = 0$ or $b_1 = \cdots = b_n = 0$ or there is $x \in \mathbb{R}$ such that $b_1 = a_1x$, …, $b_n = a_nx$.*

Solution. We have $\sqrt{\sum_{i=1}^n (a_i + b_i)^2} \le \sqrt{\sum_{i=1}^n a_i^2} + \sqrt{\sum_{i=1}^n b_i^2}$ iff

$$\left(\sqrt{\sum_{i=1}^n (a_i + b_i)^2}\right)^2 \le \left(\sqrt{\sum_{i=1}^n a_i^2} + \sqrt{\sum_{i=1}^n b_i^2}\right)^2,$$

which is equivalent to

$$\sum_{i=1}^n (a_i + b_i)^2 \le \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 + 2\sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2},$$

to

$$\sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 + 2\sum_{i=1}^n a_ib_i \le \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 + 2\sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2},$$

and to

$$\sum_{i=1}^n a_ib_i \le \sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2},$$

which follows from the Cauchy-Schwarz inequality in (b). Moreover, all the inequalities are "<" (*strict*, or *sharp*), only if all $a_i = 0$ or there is $x \in \mathbb{R}$ such that $b_i = a_ix$ for all $i = 1, \ldots, n$. (Notice that this last statement is not biconditional ("iff"): if $b_i = a_ix$, $i = 1, \ldots, n$, with $x < 0$, the inequalities are still strict.)

Chapter 2, pp. 27-33:

1(ii). *Prove by induction:*

10pt (i) *For all* n, $1^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution. The equality holds for $n = 1$: $1^2 = \frac{1(1+1)(2+1)}{6}$. Assume it holds for some $n \in \mathbb{N}$. Then for $n+1$ it also holds, since

$$1^2 + \ldots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)}{6}(2n^2 + n + 6n + 6) = \frac{(n+1)}{6}(n+2)(2n+3) = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}.$$

Hence, by induction, it holds for all $n \in \mathbb{N}$.

10pt (ii) *For all* n, $1^3 + \ldots + n^3 = (1 + \ldots + n)^2$.

Solution. The equality clearly holds for $n = 1$. Assume it holds for some $n \in \mathbb{N}$. Then it also holds for $n+1$, since

$$1^3 + \ldots + n^3 + (n+1)^3 = (1 + \ldots + n)^2 + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 = \frac{(n+1)^2}{4}\left(n^2 + 4(n+1)\right) = \frac{(n+1)^2}{4}(n+2)^2 = \left(\frac{(n+1)(n+2)}{2}\right)^2 = (1 + \ldots + n + (n+1))^2.$$

By induction, the equality holds for all $n \in \mathbb{N}$.

Cf. 23. *Prove (by induction) that for any* $a, b \in \mathbb{R}$ *and* $n, m \in \mathbb{N}$

5pt (i) $(a^n)^m = a^{nm}$,

Solution. Let $n \in \mathbb{N}$. We have $(a^n)^1 = a^n = a^{n1}$, and, if $(a^n)^m = a^{nm}$ for some m, then

$$(a^n)^{m+1} = (a^n)^m a^n = a^{nm} a^n = a^{nm+n} = a^{n(m+1)}.$$

So, by induction on m, the equality holds for all m (and all n).

5pt (ii) $(ab)^n = a^n b^n$,

Solution. $(ab)^1 = ab = a^1 b^1$ is true. If for some n, $(ab)^n = a^n b^n$, then $(ab)^{n+1} = (ab)^n (ab) = (a^n b^n)(ab) = (a^n a)(b^n b) = a^{n+1} b^{n+1}$; so, by induction, the equality holds for all n.

5pt (iii) $(a^{-1})^n = (a^n)^{-1}$.

Solution. By (ii), $a^n (a^{-1})^n = (aa^{-1})^n = 1^n = 1$, so $(a^{-1})^n = (a^n)^{-1}$.

5pt **A3.** (a) *Prove that for all integer* $n \geq 3$, $2^n > 2n + 1$.

Solution. For $n = 3$, $2^n = 8 > 7 = 2n + 1$. If $2^n > 2n + 1$ for some $n \geq 3$, then $2^{n+1} = 2 \cdot 2^n = 2^n + 2^n > 2 + (2n+1) = 2(n+1) + 1$. So, by induction, $2^n > 2n + 1$ for all $n \geq 3$.

5pt (b) *Prove that for all integer* $n \geq 5$, $2^n > n^2$.

Solution. For $n = 5$, $2^5 = 32 > 25 = n^2$. If $2^n > n^2$ for some $n \geq 5$, then $2^{n+1} = 2 \cdot 2^n = 2^n + 2^n$ and $(n+1)^2 = n^2 + 2n + 1$; since $2^n > n^2$ and $2^n > 2n + 1$ by (a), we obtain that $2^{n+1} > (n+1)^2$. So, by induction, $2^n > n^2$ for all $n \geq 5$.