Math 4181H **Solutions to Homework 3**

10pt **A1.** Prove that for any $n, k \in \mathbb{N}$ with $k \le n$, $\binom{n}{k}$ equals the number of k-element subsets in an n-element set: if X is a set with $|X| = n$, then $\binom{n}{k} = \#\{A \subseteq X : |A| = k\}$.

Solution. The statement is true for $n = 1$: we have $\binom{1}{1} = 1$ and a 1-element set contains exactly one subset of cardinality 1. Assume, by induction, that the statement is true for some $n \in \mathbb{N}$. Let X be a $(n+1)$-element set. Pick an element x_0 of X and put $X' = X \setminus \{x_0\}$, then $|X'| = n$. The k-element subsets of X are of two sorts: those that contain x_0, and those that don't. For any $A \subseteq X$ such that $|A| = k$ and $x_0 \in A$ put $A' = A \setminus \{x_0\}$, then $A' \subseteq X'$ and $|A'| = k - 1$; conversely, for any $A' \subseteq X'$ with $|A'| = k - 1$ for $A = A' \cup \{x_0\}$ we have $A \subseteq X$ and $|A| = k$. So, the set of k-element subsets of X that contain x_0 is in one-to-one correspondence with the set of $(k-1)$-element subsets of X'. So, by induction hypothesis, $\#\{A \subseteq X : |A| = k \wedge x_0 \in A\} = \binom{n}{k-1}$. As for k-element sibsets of X that don't contain x_0, these are exactly k-element sybsets of X', so, by induction hypothesis, $\#\{A \subseteq X : |A| = k \wedge x_0 \notin A\} = \binom{n}{k}$. The total number of k-element subsets of X is therefore $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$, which justifies the induction step.

Chapter 2, pp. 27-33:

10pt **3.** (e) *Prove that for any* $n \in \mathbb{N}$*:*

(i) $\sum_{j=0}^{n} \binom{n}{j} = 2^n$.

Solution. By the binomial formula, $2^n = (1+1)^n = \sum_{j=0}^{n} \binom{n}{j} 1^j 1^{n-j} = \sum_{j=0}^{n} \binom{n}{j}$.

(ii) $\sum_{j=0}^{n} (-1)^j \binom{n}{j} = 0$.

Solution. By the binomial formula, $0 = 0^n = (-1+1)^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j 1^{n-j} = \sum_{j=0}^{n} (-1)^j \binom{n}{j}$.

(iii, iv) $\sum_{\substack{0 \le j \le n \\ j \text{ is odd}}} \binom{n}{j} = 2^{n-1}$ and $\sum_{\substack{0 \le j \le n \\ j \text{ is even}}} \binom{n}{j} = 2^{n-1}$.

Solution. Let $E = \sum_{\substack{0 \le j \le n \\ j \text{ is even}}} \binom{n}{j}$ and $O = \sum_{\substack{0 \le j \le n \\ j \text{ is odd}}} \binom{n}{j}$. From (ii) we have

$$0 = \sum_{j=0}^{n} (-1)^j \binom{n}{j} = \sum_{\substack{0 \le j \le n \\ j \text{ is odd}}} (-1)^j \binom{n}{j} + \sum_{\substack{0 \le j \le n \\ j \text{ is even}}} (-1)^j \binom{n}{j} = 1 \sum_{\substack{0 \le j \le n \\ j \text{ is even}}} \binom{n}{j} + (-1) \sum_{\substack{0 \le j \le n \\ j \text{ is odd}}} \binom{n}{j} = E - O,$$

so $E = O$. From (i) we have $E + O = 2^n$, so $O = E = 2^n/2 = 2^{n-1}$.

10pt **A2.** *Prove that for any* $m \in \mathbb{N}$ *and* $n \in \mathbb{Z}$ *there exist* $k, r \in \mathbb{Z}$ *with* $0 \le r < m$ *such that* $n = km + r$.

Solution. First, assume that $n \in \mathbb{N}$, and use complete induction on n. If $n < m$, then $n = 0m + n$ and we are done. If $n > m$, let $n' = n - m$, then $n' \in \mathbb{N}$ and $n' < n$; by induction hypothesis, $n' = k'm + r$ for some $k' \in \mathbb{Z}$ and $r \in \mathbb{Z}$ with $0 \le r < m$, so, $n = n' + m = k'm + r + m = (k'+1)m + r$.

If $n = 0$, then $n = 0m + 0$.

If $n < 0$, then, as we proved, $-n = lm + s$ for some $l \in \mathbb{Z}$ and $s \in \mathbb{Z}$ with $0 \le s < m$. If $s = 0$, then $n = (-l)m + 0$. If $s \ge 1$, then $n = (-l)m - s = (-l-1)m + (m-s)$, where $0 < m - s < m$.

Another solution. Let $k = [n/m]$ and $z = \{n/m\}$ (the integer and the fractional parts of n/m), then $n/m = k + z$, so $n = km + zm$. Put $r = zm$, then $n = km + r$. Since $n, km \in \mathbb{Z}$, $r = n - km \in \mathbb{Z}$ as well. Since $0 \le z < 1$, $0 \le r < m$.

5pt **A3.** (b) *Show (by example) that if* $a, b \in \mathbb{R}$ *are irrational, then* $a + b$ *can be rational or irrational. Prove that if* a *is irrational and* b *is rational, then* $a + b$ *is irrational.*

Solution. $\pm\sqrt{2}$ is irrational, $\sqrt{2} + (-\sqrt{2}) = 0$ is rational, $\sqrt{2} + \sqrt{2} = 2\sqrt{2}$ is irrational.

If b is rational and $a + b$ is rational, then $a = (a+b) - b$ is rational. (So, if b is rational and a is irrational, then $a + b$ cannot be rational.)

5pt (c) *Show (by example) that if* $a, b \in \mathbb{R}$ *are irrational, then* ab *can be rational or irrational. Prove that if* a *is irrational and* $b \ne 0$ *is rational, then* ab *is irrational.*

Solution. $\sqrt{2}$ and $\sqrt{3}$ are irrational, $\sqrt{2}\sqrt{3} = \sqrt{6}$ is irrational, $\sqrt{2}\sqrt{2} = 2$ is rational.

If $b \ne 0$ is rational and ab is rational, then $a = (ab)b^{-1}$ is rational. (So, if $b \ne 0$ is rational and a is irrational, then ab cannot be rational.)

5pt (a) *If $a \in \mathbb{R}$ is irrational, prove that a^{-1} is irrational.*

Solution. First of all, since a is irrational, $a \neq 0$, so a^{-1} exists. Proving by contraposition, if a^{-1} is rational, $a^{-1} = n/m$ with $n, m \in \mathbb{Z}$, $n \neq 0$, then $a = m/n$ and so, is also rational. Hence, if a is irrational, then a^{-1} is irrational.

Or, using (c): if a is irrational and a^{-1} is rational, then $1 = aa^{-1}$ is irrational, contradiction.

5pt (d) *Prove that if $a > 0$ is irrational, then $\sqrt{a}$ is irrational. Show (by example) that if $a \in \mathbb{R}$ is irrational, then a^2 can be rational or irrational.*

Solution. If $a > 0$ and $\sqrt{a}$ is rational, then $a = (\sqrt{a})^2$ is rational; so, if a is irrational, then $\sqrt{a}$ cannot be rational.

$(\sqrt{2})^2 = 2$ is rational. If b is irrational (say, $b = \sqrt{2}$), then $a = \sqrt{b}$ is irrational with $a^2 = b$ being also irrational.

5pt **14(b).** *Prove that $\alpha = \sqrt{2} + \sqrt{3}$ is irrational.*

Solution. We have $\alpha^2 = 5 + 2\sqrt{6}$, which is irrational. ($\sqrt{6}$ is irrational, so $2\sqrt{6}$ is irrational, so $5 + 2\sqrt{6}$ is irrational.) Hence, α is also irrational.

5pt **A4.** *Prove that the set of irrational numbers is dense in $\mathbb{R}$.*

Solution. The numbers of the form $\sqrt{2} + q$ with $q \in \mathbb{Q}$ are all irrational. Let's show that numbers of this form are dense in $\mathbb{R}$. Take any (open) interval (a, b), find a rational number q in the interval $\big(a - \sqrt{2}, b - \sqrt{2}\big)$, then $q + \sqrt{2}$ is in the interval (a, b).

Another solution. Every interval I in $\mathbb{R}$ has cardinality of continuum, whereas $\mathbb{Q}$ is countable, so I cannot consist of points of $\mathbb{Q}$ only. Hence, every interval in $\mathbb{R}$ contains irrational numbers.

5pt **A5.** *Let $A = \{a \in \mathbb{Q} \mid a^2 < 2\}$. Find $\sup A$ and $\inf A$ (and prove your statement, of course).*

Solution. I claim that $\sup A = \sqrt{2}$. Indeed, for any $a \in A$ we have $a < 0$ or $a > 0$, $a^2 < 2 = \sqrt{2}^2$; in both cases we have $a < \sqrt{2}$, that is, $\sqrt{2}$ is an upper bound of A. And if $c < \sqrt{2}$, then the interval $(c, \sqrt{2})$ contains a positive rational number a, then $a^2 < \sqrt{2}^2 = 2$, so $a \in A$ and $a > c$; hence, c is not an upper bound of A.

Similarly, $\inf A = -\sqrt{2}$. Or, just notice that $-A = A$, so $\inf A = -\sup(-A) = -\sup A = -\sqrt{2}$.

5pt **A6.** (a) *For a nonempty set $A \subseteq \mathbb{R}$ and a number $c \in \mathbb{R}$ define $cA = \{ca,\ a \in A\}$. Prove that if $c > 0$, then $\sup(cA) = c \sup A$, and if $c < 0$, then $\sup cA = c \inf A$.*

Solution. I'll use "the criterion for supremum" which works in any case, regardless of whether the supremum is finite or infinite: $b = \sup A$ *iff $a \leq b$ for all $a \in A$ and for any $d < b$ there is $a \in A$ such that $a > d$.* The corresponding criterion for infimum is: $b = \inf A$ *iff $a \geq b$ for all $a \in A$ and for any $d > b$ there is $a \in A$ such that $a < d$.*

Ok, let $c > 0$, and let $b = \sup A$, finite or infinite. Since $b \geq a$ for all $a \in A$, we have $cb \geq ca$ for all $a \in A$. For any $d < cb$ we have $d/c < b$, so there is $a \in A$ such that $a > d/c$, so $ca > d$, so d is not an upper bound of cA. Hence, $cb = \sup(cA)$.

Now let $c < 0$, and let $b = \inf A$, finite or infinite. Since $b \leq a$ for all $a \in A$, we have $cb \geq ca$ for all $a \in A$. (Multiplication or division by c reverses the order on $\mathbb{R}$.) For any $d < cb$ we have $d/c > b$, so there is $a \in A$ such that $a < d/c$, so $ca > cd$, so cd is not an upper bound of cA. Hence, $cb = \sup(cA)$.

10pt (b) *Let A and B be nonempty subsets of $(0, +\infty)$ (that is, $a, b > 0$ for all $a \in A$ and $b \in B$). Let $AB = \{ab : a \in A,\ b \in B\}$. Prove that $\sup(AB) = \sup A \sup B$.*

Solution. Let $s = \sup A$ and $r = \sup B$, then $s, r > 0$. For any $c \in AB$ we have $c = ab$ for some $a \in A$ and $b \in B$; since $0 < a \leq s$ and $0 < b \leq r$ we have $c = ab \leq sr$.

Let's assume that A and B are bounded above, so $s, r \in \mathbb{R}$. Now let $\varepsilon > 0$ be given. Let $0 < \delta < \min\{s, r\}$, to be defined later. Choose $a \in A$ with $a > s - \delta$ and $b \in B$ with $b > r - \delta$, then $ab > (s - \delta)(r - \delta) = sr - (s + r)\delta + \delta^2 > sr - (s + r)\delta$. So, if $\delta < \varepsilon/(s + r)$, we obtain that $ab > sr - \varepsilon$, with $ab \in AB$. Hence, $sr = \sup(AB)$.

If at least one of A and B, say A, is unbounded above, then $s = +\infty$, thus $sr = +\infty$. In this case the set AB is also unbounded above and $\sup(AB) = +\infty$ as well.