Math 4181H **Solutions to Homework 4**

5pt **A1.** *If (x_n) and (y_n) are two sequence in $\mathbb{R}$ converging to the same limit a, prove that the sequence $(x_1, y_1, x_2, y_2, \ldots)$ also converges to a.*

Solution. Let $\varepsilon > 0$. Find k such that for all $n \geq k$, $|x_n - a| < \varepsilon$ and $|y_n - a| < \varepsilon$. Then for any $n \geq 2k$, if n is odd, then the n-th element of the sequence is $z_n = x_{(n+1)/2}$ with $(n+1)/2 > k$, so $|z_n - a| < \varepsilon$; and if n is even, then the n-th element of the sequence is $z_n = y_{n/2}$ with $n/2 \geq k$, so $|z_n - a| < \varepsilon$ as well.

Chapter 22, pp. 460-465:

1. *Prove that*

10pt (iii) $\lim\big(\sqrt[8]{n^2+1} - \sqrt[4]{n+1}\big) = 0$.

Solution. First, we have

$$\big|\sqrt{n^2+1} - (n+1)\big| = \Big|\frac{n^2+1-(n+1)^2}{\sqrt{n^2+1}+(n+1)}\Big| < \frac{2n}{2n+1} < 1$$

for all n. Next,

$$\big|\sqrt[4]{n^2+1} - \sqrt{n+1}\big| = \Big|\frac{\sqrt{n^2+1}-(n+1)}{\sqrt[4]{n^2+1}+\sqrt{n+1}}\Big| < \Big|\frac{1}{\sqrt[4]{n^2+1}+\sqrt{n+1}}\Big| \longrightarrow 0,$$

so $\sqrt[4]{n^2+1} - \sqrt{n+1} \longrightarrow 0$. Finally,

$$\sqrt[8]{n^2+1} - \sqrt[4]{n+1} = \frac{\sqrt[4]{n^2+1} - \sqrt{n+1}}{\sqrt[8]{n^2+1} + \sqrt[4]{n+1}} \longrightarrow 0.$$

(The numerator tends to 0 and the denominator to ∞.)

5pt (iv) $\lim(n!/n^n) = 0$.

Solution. For any even n we have

$$0 < \tfrac{n!}{n^n} = \tfrac{1\cdot 2\cdot 3\cdots(n-1)\cdot n}{n\cdot n\cdot n\cdots n\cdot n} = \big(\tfrac{1}{n}\cdot\tfrac{2}{n}\cdot\tfrac{3}{n}\cdots\tfrac{n-1}{n}\cdot\tfrac{n}{n}\big) \leq \tfrac{1}{n}\cdot 1\cdot 1\cdots 1 = \tfrac{1}{n}.$$

Since $\frac{1}{n}^n \longrightarrow 0$, $\lim\frac{n!}{n^n} = 0$ by the squeeze theorem.

5pt (vii) $\lim\sqrt[n]{n^2+n} = 1$.

Solution. For any $n \in \mathbb{N}$ we have $n^2 < n^2+n < 2n^2$ and so, $\sqrt[n]{n^2} < \sqrt[n]{n^2+n} < \sqrt[n]{2n^2}$. Now, $\sqrt[n]{n} \longrightarrow 1$ and $\sqrt[n]{2} \longrightarrow 1$, so $\sqrt[n]{n^2} = \big(\sqrt[n]{n}\big)^2 \longrightarrow 1$ and $\sqrt[n]{2n^2} = \sqrt[n]{2}\big(\sqrt[n]{n}\big)^2 \longrightarrow 1$. By the squeeze theorem, $\lim\sqrt[n]{n^2+n} = 1$.

5pt (viii) *For any $a, b \geq 0$*, $\lim\sqrt[n]{a^n+b^n} = \max\{a,b\}$.

Solution. Without loss of generality, let $a \geq b$. Then for any n, $a = \sqrt[n]{a} \leq \sqrt[n]{a^n+b^n} \leq \sqrt[n]{a^n+a^n} = a\sqrt[n]{2}$. Since $a\sqrt[n]{2} \longrightarrow a$, by the squeeze theorem, $\lim\sqrt[n]{a^n+b^n} = a$.

2. *Find the following limits*

5pt (ii) $\lim\big(n - \sqrt{n+a}\sqrt{n+b}\big)$.

Solution.

$$n - \sqrt{n+a}\sqrt{n+b} = \tfrac{n^2-(n+a)(n+b)}{n+\sqrt{n+a}\sqrt{n+b}} = -\tfrac{(a+b)n}{n+\sqrt{n+a}\sqrt{n+b}} - \tfrac{ab}{n+\sqrt{n+a}\sqrt{n+b}} = -\tfrac{a+b}{1+\sqrt{1+a/n}\sqrt{1+b/n}} - \tfrac{ab}{n+\sqrt{n+a}\sqrt{n+b}} \longrightarrow -\tfrac{a+b}{2} - 0 = -\tfrac{a+b}{2}.$$

5pt (v) $\lim\frac{a^n-b^n}{a^n+b^n}$.

Solution. If $|a| > |b|$, then $\frac{a^n-b^n}{a^n+b^n} = \frac{1-(b/a)^n}{1+(b/a)^n} \longrightarrow 1$. If $|a| < |b|$, then $\frac{a^n-b^n}{a^n+b^n} = \frac{(a/b)^n-1}{(a/b)^n+1} \longrightarrow -1$. If $a = b$, then the limit is 0.

5pt (vii) $\lim\frac{2^{n^2}}{n!}$.

Solution. For all n, $2^{n^2} = (2^n)^n > n^n$, and $\lim(n!/n^n) = 0$ by 1(iv), so $\lim(n!/2^{n^2}) = 0$ by the squeeze theorem, so $\lim\frac{2^{n^2}}{n!} = \infty$.

5pt **5.** (a) *If* $0 < a < 2$*, prove that* $a < \sqrt{2a} < 2$.

Solution. $0 < a < 2$ implies that $0 < a^2 < 2a < 4$, so $a < \sqrt{2a} < 2$.

5pt (b) *Prove that the sequence* $x_1 = \sqrt{2}$, $x_2 = \sqrt{2\sqrt{2}}$, $x_3 = \sqrt{2\sqrt{2\sqrt{2}}}$, *…, converges.*

Solution. We have $x_{n+1} = \sqrt{2x_n}$ for all $n \in \mathbb{N}$. Since $0 < x_1 < 2$, by (a) and by induction, $x_n < x_{n+1} < 2$ for all n. Thus (x_n) is a bounded increasing sequence, and so, converges.

5pt (c) *Find* $\lim x_n$.

Solution. Let $a = \lim x_n$ (it exists by (b)). Then

$$a = \lim x_{n+1} = \lim \sqrt{2x_n} = \sqrt{2a},$$

so $a^2 = 2a$, and since $a > 0$, $a = 2$.

(Ok, I used the fact that if $x_n \longrightarrow a$ then $\sqrt{x_n} \longrightarrow \sqrt{a}$. If we don't know it, we can write: $x_{n+1}^2 = 2x_n$ for all n, so, taking the limits of both parts, we get that $a^2 = 2a$, so $a = 2$.)

6. *Let* $0 < a_1 < b_1$ *and define* $a_{n+1} = \sqrt{a_n b_n}$ *and* $b_{n+1} = \frac{1}{2}(a_n + b_n)$.

5pt (a) *Prove that* (a_n) *increases,* (b_n) *decreases, and both converge.*

Solution. For each n, $b_n > a_n$ by the arithmetic/geometric mean inequality. It follows that, for each n, $a_{n+1} = \frac{1}{2}(a_n + b_n) > \frac{1}{2}(a_n + a_n) = a_n$ and $b_{n+1} = \sqrt{a_n b_n} < \sqrt{b_n b_n} = b_n$, that is, (a_n) is (strictly) increasing and (b_n) is (strictly) decreasing. Also, the sequence (a_n) is bounded above (by b_1), and the sequence (b_n) is bounded below (by a_1). So, they both converge.

5pt (b) *Prove that they have the same limit.*

Solution. Let $a = \lim a_n$ and $b = \lim b_n$. Then $b = \lim b_{n+1} = \lim \frac{1}{2}(a_n + b_n) = \frac{1}{2}(a + b)$, so $a = b$.

10pt **7(b).** *Find* $1 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\dots}}}$*, that is: Let* $x_1 = 1$ *and* $x_{n+1} = 1 + \frac{1}{1+x_n}$, $n \in \mathbb{N}$*; prove that the sequence* (x_n) *converges and find its limit.*

Solution. By induction, $x_n > 0$ and so ≥ 1 for all n. For any n,

$$x_{n+2} - x_{n+1} = \Big(1 + \frac{1}{1 + x_{n+1}}\Big) - \Big(1 + \frac{1}{1 + x_n}\Big) = \frac{x_n - x_{n+1}}{(1 + x_{n+1})(1 + x_n)}.$$

Since $x_n, x_{n+1} \geq 1$, we have $0 < \frac{1}{(1+x_{n+1})(1+x_n)} \leq \frac{1}{4}$, so $|x_{n+2} - x_{n+1}| \leq \frac{1}{4}|x_{n+1} - x_n|$. By a theorm proved in class (or by exercise 22), the sequence (x_n) is Cauchy and so, converges.

Now let $a = \lim_{n\to\infty} x_n$, then $a \geq 1$. We have

$$a = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\Big(1 + \frac{1}{1 + x_n}\Big) = 1 + \frac{1}{1 + \lim_{n\to\infty} x_n} = 1 + \frac{1}{1 + a},$$

so $a + a^2 = 1 + a + 1$, so $a^2 = 2$, and since $a \geq 1$, we get $a = \sqrt{2}$.