Math 4181H **Solutions to Homework 5**

5pt **A1.** *Let (x_n) be a sequence in $\mathbb{R}$. Prove that if (x_n) doesn't diverge to ∞, then it has at least one limit point.*

Solution. If (x_n) doesn't diverge to ∞, then there exists M such that for any k there is $n \geq k$ such that $|x_n| \leq M$. (That is, the set $\{n : |x_n| \leq M\}$ is infinite.) Thus, we can construct a subsequence of (x_n) that is bounded by M: choose n_1 such that $|x_{n_1}| \leq M$, then choose $n_2 > n_1$ such that $|x_{n_2}| \leq M$, etc. Since the sequence (x_{n_i}) is bounded, by the Bolzano-Weierstrass theorem it has a limit point, which is a limit point of (x_n) too.

A2. *Let (x_n) be a sequence in $\mathbb{R}$.*

5pt (a) *Prove that* $\limsup x_n = +\infty$ *iff* (x_n) *is unbounded above.*

Solution. The sequence $s_k = \sup\{x_n,\ n \geq k\}$, $k = 1, 2, \ldots$, is decreasing, and $\limsup x_n = \lim s_k$. If $lims_k = +\infty$ then in particular $s_1 = \sup\{x_n,\ n \in \mathbb{N}\} = +\infty$, that is, the sequence (x_n) is unbounded above. If (x_n) is bounded above by $M \in \mathbb{R}$, then $s_k \leq M$ for all k, so $\lim s_k \leq M$.

5pt (b) *Prove that* $\limsup x_n = -\infty$ *iff* $x_n \longrightarrow -\infty$.

Solution. For every n, $x_n \leq s_n$; so if $\limsup x_n = \lim s_k = -\infty$, then $\lim x_n = -\infty$ by "comparison principle". Conversely, if $x_n \longrightarrow -\infty$, for any $M \in \mathbb{R}$ there is k such that $x_n < M$ for all $n \geq k$, then $s_k \leq M$ and $s_n \leq M$ for all $n \geq k$; thus $\lim s_k = -\infty$.

5pt **A3.** *Let (x_n) be a sequence in $\mathbb{R}$; prove that (x_n) has limit (finite or infinite) iff* $\limsup x_n = \liminf x_n$, *in which case* $\lim x_n = \limsup x_n$.

Solution. Suppose $a = \lim x_n$ exists. Then for any $b > a$ the set $\{n : x_n > b\}$ is finite, so $\limsup x_n \leq a$; and for any $b < a$ the set $\{n : x_n < b\}$ is finite, so $\liminf x_n \geq a$; hence, $\limsup x_n = \liminf x_n = a$.

Conversely, suppose that $a = \limsup x_n = \liminf x_n$. If $a \in \mathbb{R}$, for any $\varepsilon > 0$ the sets $\{n : x_n \geq a + \varepsilon\}$ and $\{n : x_n \leq a - \varepsilon\}$ are finite, so $|x_n - a| < \varepsilon$ for all but finitely many n; hence, $a = \lim x_n$. If $a = +\infty$, then for any $M \in \mathbb{R}$ the set $\{n : x_n < M\}$ is finite, so $\lim x_n = +\infty$. If $a = -\infty$, then for any $M \in \mathbb{R}$ the set $\{n : x_n > M\}$ is finite, so $\lim x_n = -\infty$.

10pt **A4.** *Let (x_n) and (y_n) be two sequences in $\mathbb{R}$ and suppose (y_n) converges. Prove that* $\limsup(x_n + y_n) = \limsup x_n + \lim y_n$.

Solution. First, assume that $s = \limsup x_n \in \mathbb{R}$. Let $a = \lim y_n$. Let $\varepsilon > 0$; there is k such that for all $n \geq k$, $x_n < s + \varepsilon/2$ and $y_n < a + \varepsilon/2$, and then $x_n + y_n < a + s + \varepsilon$. Also, find m such that $x_n > a - \varepsilon/2$ for all $n \geq m$; then for any k there is $n \geq \max\{k, m\}$ such that $y_n > a - \varepsilon/2$, and then $x_n + y_n > a + s - \varepsilon$. Hence, $a + s = \limsup(x_n + y_n)$.

We have $\limsup x_n = +\infty$ iff (x_n) is unbounded above, then, since (x_n) is bounded, $(x_n + y_n)$ is unbounded above, and $\limsup(x_n + y_n) = +\infty = \limsup x_n + \lim y_n$.

We have $\limsup x_n = -\infty$ iff $(x_n) \longrightarrow -\infty$, in which case $\limsup(x_n + y_n) = \lim(x_n + y_n) = -\infty = \limsup x_n + \lim y_n$.

A5. *Let (x_n) be a sequence in $\mathbb{R}$.*

10pt (a) *Prove that* $\limsup \frac{x_1+\cdots+x_n}{n} \leq \limsup x_n$. *(Similarly,* $\liminf \frac{x_1+\cdots+x_n}{n} \geq \liminf x_n$*; don't prove it.)*

Solution. Let $b > \limsup x_n$. Find k such that $x_n < b$ for all $n > k$. Then for any $n > k$,

$$\frac{x_1 + \cdots + x_n}{n} = \frac{x_1 + \cdots + x_k}{n} + \frac{x_{k+1} + \cdots + x_n}{n} < \frac{x_1 + \cdots + x_k}{n} + \frac{b(n-k)}{n} = \frac{x_1 + \cdots + x_k - bk}{n} + b.$$

Taking the lim sup of both parts we get that

$$\limsup \frac{x_1 + \cdots + x_n}{n} \leq \limsup\Big(\frac{x_1 + \cdots + x_k - bk}{n} + b\Big) = \lim\Big(\frac{x_1 + \cdots + x_k - bk}{n} + b\Big) = b.$$

Since this is true for all $b > \limsup x_n$, we obtain that $\limsup \frac{x_1+\cdots+x_n}{n} \leq \limsup x_n$.

Another solution. Put $a_n = x_1 + \cdots + x_n$, $n \in \mathbb{N}$, and $b_n = n$, $n \in \mathbb{N}$. For every n we then have $a_{n+1} - a_n = x_n$ and $b_{n+1} - b_n = 1$. Since the sequence (b_n) is increasing and tends to $+\infty$, Stolz's theorem applies to sequences (a_n) and (b_n), which says that $\limsup \frac{a_n}{n} \leq \limsup \frac{x_n}{1} = \limsup x_n$ (and $\liminf \frac{a_n}{n} \geq \liminf x_n$).

5pt (b) *Prove that if* $\lim x_n$ *exists, then* $\lim \frac{x_1+\cdots+x_n}{n}$ *also exists and equals* $\lim x_n$.

Solution. Let $\lim x_n = a$, then by (a) $a \leq \liminf \frac{x_1+\cdots+x_n}{n} \leq \limsup \frac{x_1+\cdots+x_n}{n} \leq a$, so $\liminf \frac{x_1+\cdots+x_n}{n} = \limsup \frac{x_1+\cdots+x_n}{n} = a$. so $\lim \frac{x_1+\cdots+x_n}{n} = a$.

Chapter 5, pp. 108-112:

5pt **8.** (a) *If a finite* $\lim_{x\to a} f(x)$ *and a finite* $\lim_{x\to a} g(x)$ *do not exist, can a finite* $\lim_{x\to a}(f(x)+g(x))$ *exist? Can a finite* $\lim_{x\to a} f(x)g(x)$ *exist?*

Solution. Yes for both. For example, take any $a \in \mathbb{R}$ and a function $f > 1$ for which $\lim_{x\to a} f(x)$ does not exist, and put $g = -f$, or $g = 1/f$ respectively.

5pt (b) *If a finite* $\lim_{x\to a} f(x)$ *and a finite* $\lim_{x\to a}(f(x) + g(x))$ *exist, must a finite* $\lim_{x\to a} g(x)$ *exist?*

Solution. Yes, since $g = (f + g) - f$.

5pt (c) *If a finite* $\lim_{x\to a} f(x)$ *exists and a finite* $\lim_{x\to a} g(x)$ *does not exist, can a finite* $\lim_{x\to a}(f(x) + g(x))$ *exist?*

Solution. No, by (b).

5pt (d) *If finite* $\lim_{x\to a} f(x)$ *and finite* $\lim_{x\to a} f(x)g(x)$ *exist, does it follow that* $\lim_{x\to a} g(x)$ *exists?*

Solution. No. Take any function g for which this limit does not exist and put $f = 0$.

5pt **9.** *Prove that* $\lim_{x\to a} f(x) = \lim_{h\to 0} f(a + h)$ *(that is, one limit exists iff the other exists, and they are equal). Only consider the case of finite limits.*

Solution. Let $\mathrm{Dom}(f) = A$. Let $\lim_{x\to a} f(x) = b$, and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that for any $x \in A \setminus \{a\}$ with $|x - a| < \delta$ one has $|f(x) - b| < \varepsilon$. Then for any $h \in A - a$ with $0 < |h| < \delta$ we have $|(a + h) - a| < \delta$, so $|f(a + h) - b| < \varepsilon$. Ths means that $\lim_{h\to 0} f(a + h) = b$.

Conversely, assume that $\lim_{h\to 0} f(a + h) = b$. Let $\varepsilon > 0$. Find $\delta > 0$ such that for any $h \in A - a$ with $0 < |h| < \delta$ one has $|f(a + h) - b| < \varepsilon$. Then for any $x \in A$ such that $0 < |x - a| < \delta$, we have $|f(x) - b| = (f(a + (x - a)) - b| < \varepsilon$, which proves that $\lim_{x\to a} f(x) = b$.

Another solution. The function $g(h) = f(a+h)$ is the composition of two functions: $\varphi(h) = a+h$, $h \in \mathrm{Dom}(f)-a$, and f, so the theorem about the limit of the composition could be used. We have $\lim_{h\to 0} \varphi(h) = a$, so, if $\lim_{x\to a} f(x)$ exists, $\lim_{h\to 0} f(a + h) = \lim_{h\to 0} f(\varphi(h)) = \lim_{x\to a} f(x)$.

Conversely, f is the composition of $\psi(x) = x - a$ and g, $f(x) = g(a - h)$, so if $\lim_{h\to 0} f(a+h) = \lim_{h\to 0} g(h)$ exists, then $\lim_{x\to a} f(x) = \lim_{h\to 0} f(a + h)$.

5pt **14(a).** *Prove that if* $\lim_{x\to 0} f(x)/x = l \in \mathbb{R}$ *and* $b \neq 0$*, then* $\lim_{x\to 0} f(bx)/x = bl$.

Solution. Given any $\varepsilon > 0$, find $\delta > 0$ such that $\left|\frac{f(x)}{x} - l\right| < \frac{\varepsilon}{|b|}$ for all $x \in \mathbb{R} \setminus \{0\}$ such that $|x| < \delta$, then for any $x \in \mathbb{R} \setminus \{0\}$ such that $|x| < \delta/|b|$ we have $bx \neq 0$ and $|bx| < \delta$, so $\left|\frac{f(bx)}{bx} - l\right| < \frac{\varepsilon}{|b|}$, so $\left|\frac{f(bx)}{x} - bl\right| < \varepsilon$.

Another solution. The function $f(bx)/(bx)$ is the composition of two functions: $x \mapsto bx$ and $g(y) = f(y)/y$. We have $\lim_{x\to 0}(bx) = b \lim_{x\to 0} x = 0$ and $bx \neq 0$ for all $x \neq 0$, so by the theorem about the limit of a composition, $\lim_{x\to 0} f(bx)/(bx) = \lim_{y\to 0} f(y)/y = l$. So, $\lim_{x\to 0} f(bx)/x = b \lim_{x\to 0} f(bx)/(bx) = bl$.

10pt **34.** *Prove that* $\lim_{x\to 0^+} f(1/x) = \lim_{x\to +\infty} f(x)$ *(that is, one limit exists iff the other exists, and they are equal). Only consider the case of finite limits.*

Solution. Assume that $\lim_{x\to 0^+} f(1/x) = b \in \mathbb{R}$. Let $\varepsilon > 0$. Find $\delta > 0$ such that $|f(1/x)-b| < \varepsilon$ when $0 < x < \delta$ and $1/x \in \mathrm{Dom}(f)$. Then for any $y \in \mathrm{Dom}(f)$ with $y > 1/\delta$ we have $0 < 1/y < \delta$, so $|f(1/(1/y)) - b| < \varepsilon$, so $|f(y) - b| < \varepsilon$. Hence, $\lim_{y\to +\infty} f(y) = b$.

In the other direction: Assume that $\lim_{x\to +\infty} f(x) = b$. Let $\varepsilon > 0$. Find $M > 0$ such that $|f(x) - b| < \varepsilon$ for all $x \in \mathrm{Dom}(f)$ with $x > M$. Then for any $y \in \mathrm{Dom}(f)$ with $0 < y < 1/M$ we have $1/y > M$, so $|f(1/y)-b| < \varepsilon$. Hence, $\lim_{y\to 0^+} f(1/y) = b$.

Another solution. $f(1/x)$ is the composition of two functions: $x \mapsto 1/x$ and $f(y)$. We have $\lim_{x\to 0^+} 1/x = +\infty$ and $\lim_{x\to +\infty} 1/x = 0^+$ (that is, $\lim = 0$ and $1/x > 0$ for all x in a nbhd of $+\infty$), so, by the theorem about the limit of the composition, if $\lim_{y\to +\infty} f(y)$ exists, then $\lim_{x\to 0^+} f(1/x) = \lim_{y\to +\infty} f(y)$, and if $\lim_{y\to 0^+} f(1/y)$ exists, then $\lim_{x\to +\infty} f(x) = \lim_{y\to 0^+} f(1/y)$.