Math 4181H **Solutions to Homework 6**

Chapter 8, p. 140:

6. *Let $B \subseteq \mathbb{R}$ and let A be a dense subset of B.*

5pt (b) *Prove that if f and g are continuous on B and $f(x) = g(x)$ for all $x \in A$, then $f = g$ on B (that is, $f(x) = g(x)$ for all $x \in B$).*

Solution. Let $a \in B$; there is a sequence (x_n) in A such that $x_n \longrightarrow a$, then $f(x_n) = g(x_n)$ for all n, so $f(a) = \lim f(x_n) = \lim g(x_n) = g(a)$.

5pt (c) *Prove that if f and g are continuous on B and $f(x) \geq g(x)$ for all $x \in A$, then $f \geq g$ on B (that is, $f(x) \geq g(x)$ for all $x \in B$). Can "$\geq$" be replaced by "$>$"?*

Solution. Assume that $f(x) < g(x)$ at some points x, then $f(y) < g(y)$ for all y in some neighborhood of x; but this neighborhood contains points of A, contradiction. "$\geq$" cannot be replaced by "$>$", as the following example shows: let $B = \mathbb{R}$, $A = \mathbb{R} \setminus \{0\}$, $f(x) = |x|$ and $g(x) = 0$, $x \in \mathbb{R}$; then $f(x) > g(x)$ for all $x \in A$, but $f(0) = g(0)$.

Another solution. Let $x \in B$; there is a sequence (x_n) in A such that $x_n \longrightarrow x$, then $f(x_n) \geq g(x_n)$ for every n and $f(x) = \lim_{n\to\infty} f(x_n) \geq \lim_{n\to\infty} g(x_n) = g(x)$.

Chapter 6, pp. 120-121:

5pt **4.** *Give an example of a function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ that is continuous nowhere, but $|f|$ is continuous everywhere.*

Solution. $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q} \end{cases}$. For every $a \in \mathbb{R}$, the limit $\lim_{x\to a} f(x)$ does not exist, thus f is (of second kind) discontinuous at every point. However, $|f| = 1$, so $|f|$ is continuous at all points.

5pt **5.** *For each point $a \in \mathbb{R}$ find a function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ which is continuous at a but is discontinuous at all other points of $\mathbb{R}$.*

Solution. For $a = 0$ this is the function $f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$. (Indeed, since $|f(x)| \leq |x|$ for all x, $\lim_{x\to 0} f(x) = 0$; for any $c \neq 0$, $\lim_{n\to\infty} f(x_n) = 0 \neq f(c)$ for any sequence (x_n) of irrational points that tends to c.)

For any other a, the function $f_a(x) = f(x - a)$ works.

5pt **6.** (b) *Find a function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ that is discontinuous at $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ but continuous at all other points.*

Solution. Put $f(x) = 1$ if $x \in \left\{\frac{1}{n},\ n \in \mathbb{N}\right\}$ and $f(x) = 0$ otherwise. Then f is discontinuous at all points of the form $\frac{1}{n}$, $n \in \mathbb{N}$, since for every $n \in \mathbb{N}$, $f(1/n) \neq 0$ and in every neighborhood of $1/n$ there are irrational points where f takes value 0. f is continuous at every other point $a \neq 0$ since a has a neighborhood where f is identically 0. At 0, f is discontinuous since $f(0) = 0$ but any neighborhood of 0 contains points where f is equal to 1.

5pt (a) *Find a function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ that is discontinuous at $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ but continuous at all other points.*

Solution. Put $f(x) = x$ if $x \in \left\{\frac{1}{n},\ n \in \mathbb{N}\right\}$ and $f(x) = 0$ otherwise. Then f is discontinuous at all points of the form $\frac{1}{n}$, $n \in \mathbb{N}$, since for every $n \in \mathbb{N}$, $f(1/n) \neq 0$ and in every neighborhood of $1/n$ there are irrational points where f takes value 0. At all other points f is continuous: if $a \neq 1/n$ for all n and $a \neq 0$, then a has a neighborhood where f is identically zero; if $a = 0$ then for any $\varepsilon > 0$, $|f(x)| \leq \varepsilon$ in the ε-neighborhood of a.

Chapter 7, pp. 130-132:

5pt **10.** *Suppose f and g are continuous on $[a, b]$, $f(a) < g(a)$, and $f(b) > g(b)$. Prove that $f(x) = g(x)$ for some $x \in [a, b]$.*

Solution. The function $h = f - g$ is continuous on $[a, b]$ with $h(a) < 0$ and $h(b) > 0$, so by the I.V.T., $h(x) = 0$ for some $x \in (a, b)$.

10pt **8.** *Suppose that f and g are continuous on an interval I, that $f^2 = g^2$, and that $f(x) \neq 0$ for all x. Prove that either $f = g$ or $f = -g$.*

Solution. Consider the function $h = g/f$; it is defined and continuous since $f(x) \neq 0$ for all x. For every $x \in I$ we have $h^2(x) = g^2(x)/f^2(x) = 1$, so either $h(x) = 1$ or $h(x) = -1$. But if there is a point $a \in I$ such that $h(a) = 1$ and a point $b \in I$ such that $h(b) = -1$, then by the I.V.T. there is $x \in (a, b)$ (or $x \in (b, a)$) such that $h(x) = 0$, so $g(x) = 0$, so $f(x) = 0$, contradiction. Hence, either $h(x) = 1$ for all x, in which case $f = g$, or $h(x) = -1$ for all x, in which case $f = -g$.

10pt **16.** (a) *Suppose that f is continuous on (a, b) and $\lim_{x\to a^+} f(x) = \lim_{x\to b^-} f(x) = +\infty$. Prove that f has a minimum on all of (a, b).*

Solution. Take any $c \in (a, b)$ and let $f(c) = z$. Since $\lim_{x\to a^+} f(x) = +\infty$, there exists $\delta_1 > 0$ such that $f(x) > z$ for all $x \in (a, a + \delta_1)$. Since $\lim_{x\to b^-} f(x) = +\infty$, there exists $\delta_2 > 0$ such that $f(x) > z$ for all $x \in (b - \delta_2, b)$. f is continuous on the closed bounded interval $I = [a + \delta_1, b - \delta_2]$, thus it attains its minimal value on this interval at some point $p \in I$, so that $f(p) \leq f(x)$ for all $x \in I$. Also, for any $x \in [a, b] \setminus I$, $f(x) > z = f(c) \geq f(p)$. So, $f(p)$ is the minimal value of f on the whole interval (a, b).

5pt **18.** *Suppose f is a continuous function on $\mathbb{R}$ with $f(x) > 0$ for all x, and $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = 0$. Prove that f attains its maximal value in $\mathbb{R}$.*

Solution. This problem can be reduced to problem 16 by considering the funcion $g(x) = 1/f(x)$. I'll however repeat the proof: Take any $c \in \mathbb{R}$, find M such that $0 < f(x) < f(c)$ for all x with $|x| > M$, find a point $y \in [-M, M]$ where f attains its maximal value on $[-M, M]$, then this values is maximal for f on whole $\mathbb{R}$ since $f(y) \geq f(c) > f(x)$ for all $x \notin [-M, M]$.

Chapter 8, Appendix, p. 146:

5pt **2.** (a) *If f and g are uniformly continuous on A, then so is $f + g$.*

Solution. Let $\varepsilon > 0$. Find $\delta_1 > 0$ such that $|f(x) - f(y)| < \varepsilon/2$ for any $x, y \in A$ with $|x - y| < \delta_1$, and $\delta_2 > 0$ such that $|g(x) - g(y)| < \varepsilon/2$ for any $x, y \in A$ with $|x - y| < \delta_2$. Put $\delta = \min(\delta_1, \delta_2)$. Then for any $x, y \in A$ with $|x - y| < \delta$ we have

$$\big|(f + g)(x) - (f + g)(y)\big| \leq |f(x) - f(y)| + |g(x) - g(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

5pt (b) *If f and g are uniformly continuous and bounded on A, then fg is unformly continuous on A.*

Solution. Let $M \in \mathbb{R}$ be such that $|f(x)|, |g(x)| < M$ for all $x \in A$. Let $\varepsilon > 0$. Find $\delta_1 > 0$ such that $|f(x) - f(y)| < \varepsilon/(2M)$ for any $x, y \in A$ with $|x - y| < \delta_1$, and $\delta_2 > 0$ such that $|g(x) - g(y)| < \varepsilon/(2M)$ for any $x, y \in A$ with $|x - y| < \delta_2$. Put $\delta = \min(\delta_1, \delta_2)$. Then for any $x, y \in A$ with $|x - y| < \delta$ we have

$$\begin{aligned}\big|(fg)(x) - (fg)(y)\big| = \big|f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)\big| &\leq |f(x)| \cdot |g(x) - g(y)| + |f(x) - f(y)| \cdot |g(y)| \\ &< M\varepsilon/(2M) + \varepsilon M/(2M) = \varepsilon.\end{aligned}$$

5pt (c) *Show that the conclusion of* (b) *fails if one of f, g is unbounded.*

Solution. Let $f(x) = x$, $x \in \mathbb{R}$, and $g(x)$ be "the distance to the nearest integer" function: $g(x) = \begin{cases} t \text{ for } x = n + t,\ n \in \mathbb{Z},\ t \in [0, \frac{1}{2}] \\ 1 - t \text{ for } x = n + t,\ n \in \mathbb{Z},\ t \in [\frac{1}{2}, 1]. \end{cases}$ (The function $g(x) = \sin x$ would also do the work.) Both functions are uniformly continuous on $\mathbb{R}$, and g is bounded. But fg is not uniformly continuous on $\mathbb{R}$. Indeed, take $\varepsilon = 1$. For $n \in \mathbb{N}$, put $x_n = n$ and $y_n = n + \frac{1}{n}$. Then, for $n \geq 2$,

$$f(y_n)g(y_n) - f(x_n)g(x_n) = \big(n + \tfrac{1}{n}\big)\tfrac{1}{n} - n0 = 1 + \tfrac{1}{n^2} > \varepsilon.$$

Now, given any $\delta > 0$, we can find $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$; then $|y_n - x_n| < \delta$, but $|f(y_n) - f(x_n)| > \varepsilon$.

5pt (d) *Suppose that f is uniformly continuous on A, g is uniformly continuous on B, and $f(A) \subseteq B$. Prove that $g \circ f$ is uniformly continuous on A.*

Solution. Let $\varepsilon > 0$. Find $\tau > 0$ such that $|g(u) - g(v)| < \varepsilon$ whenever $u, v \in B$, $|u - v| < \tau$. Find $\delta > 0$ such that $|f(x) - f(y)| < \tau$ whenever $x, y \in A$, $|x - y| < \delta$. Then for any $x, y \in A$ with $|x - y| < \delta$ we have $f(x), f(y) \in B$, $|f(x) - f(y)| < \tau$, so $|g(f(x)) - g(f(y))| < \varepsilon$. This proves that $g \circ f$ is uniformly continuous on A.