Math 4181H **Solutions to Homework 7**

Chapter 9, pp. 163-166:

5pt **19(a).** *Suppose that f, g and h are defined in a neighborhood of a, f and h are differentiable at a, $f(a) = h(a)$, $f'(a) = h'(a)$, and $f(x) \le g(x) \le h(x)$ or $h(x) \le g(x) \le f(x)$ for all x in a neighborhood of a. Prove that g is differentiable at a with $g'(a) = f'(a)$.*

Solution. Let $b = f(a) = h(a)$, then also $g(a) = b$. For any x in a neighborhood I of a we have $f(x) - b \le g(x) - b \le h(x) - b$ or $h(x) - b \le g(x) - b \le f(x) - b$. Thus for any $x \in I \setminus \{a\}$

$$\text{either} \quad \frac{f(x)-b}{x-a} \le \frac{g(x)-b}{x-a} \le \frac{h(x)-b}{x-a} \quad \text{or} \quad \frac{h(x)-b}{x-a} \le \frac{g(x)-b}{x-a} \le \frac{f(x)-b}{x-a}.$$

Since $\lim_{x\to a} \frac{f(x)-b}{x-a} = f'(a) = h'(a) = \lim_{x\to a} \frac{h(x)-b}{x-a}$, by the squeeze theorem $g'(a) = \lim_{x\to a} \frac{g(x)-c}{x-a}$ exists and equals $f'(a)$.

5pt **22.** (a) *Suppose f is differentiable at a. Prove that $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h}$.*

Solution. For any h we have

$$\tfrac{f(a+h)-f(a-h)}{2h} = \tfrac{1}{2}\Big(\tfrac{f(a+h)-f(a)}{h} + \tfrac{f(a)-f(a-h)}{h}\Big) = \tfrac{1}{2}\Big(\tfrac{f(a+h)-f(a)}{h} + \tfrac{f(a-h)-f(a)}{-h}\Big).$$

Since $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} = f'(a)$, $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = \frac{1}{2}(f'(a) + f'(a)) = f'(a)$. (The function $\varphi(h) = \frac{f(a-h)-f(a)}{-h}$ is the composition of the functions $h \mapsto -h$ and $\frac{f(a+h)-f(a)}{h}$; since $-h \to 0$ as $h \to 0$, by the theorem on the limit of composition, $\lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} = f'(a)$ as well.)

5pt (b) *Give an example of a function for which the limit in (a) exists and is finite, but which is not differentiable at a.*

Solution. The function $f(x) = |x|$ is such: $\frac{f(0+h)-f(0-h)}{2h} = 0$ for all $h \ne 0$, so $\lim_{h\to 0} \frac{f(0+h)-f(0-h)}{2h} = 0$, but f is not differentiable at 0.

10pt **23-24.** *Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable on $\mathbb{R}$. If f is even, prove that f' is odd, and if f is odd, prove that f' is even.*

Solution. Let f be even. Define $g(x) = f(-x)$, $x \in \mathbb{R}$. Then $g = f$, so, $g'(x) = f'(x)$ for all x. On the other hand, g is the composition of the function $x \mapsto -x$ and of f, so, by the chain rule (the theorem about the derivative of the composition), for any x, $g'(x) = f'(-x)(-1) = -f'(-x)$. Hence, for any x, $f'(x) = -f'(-x)$, that is, f' is an odd function.

Let f be odd. Define $g(x) = f(-x)$, $x \in \mathbb{R}$. Then $g = -f$, so, $g'(x) = -f'(x)$ for all x. On the other hand, g is the composition of the function $x \mapsto -x$ and of f, so, by the chain rule, for any x, $g'(x) = f'(-x)(-1) = -f'(-x)$. Hence, for any x, $-f'(x) = -f'(-x)$, so $f'(x) = f'(-x)$, that is, f' is an even function.

Chapter 10, pp. 181-187:

5pt **16.** (a) *If f is differentiable at a and $f(a) \ne 0$, prove that $|f|$ is also differentiable at a.*

Solution. W.l.o.g. assume that $f(a) > 0$. Since f is differentiable at a, f is continuous at a, so $f(x) > 0$ for all x in a neighborhood of a, so $|f(x)| = f(x)$ for all x in this neighborhood, so $|f|'(a) = \lim_{x\to a} \frac{|f(x)|-|f(a)|}{x-a} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$.

5pt (b) *Give a counterexample if $f(a) = 0$.*

Solution. The function $f(x) = x$ is differentiable at 0 whereas $|f(x)| = |x|$ is not.

5pt (c) *If f and g are differentiable at a and $f(a) \ne g(a)$, prove that* $\max\{f, g\}$ *is differentiable at a.*

Solution. If $f(a) = g(a)$, then $(f - g)(a) \ne 0$, so, by (a), $|f - g|$ is differentiable at a, so $\max\{f, g\} = \frac{1}{2}\big(f + g + |f - g|\big)$ is differentiable at a.

5pt **29.** *Suppose f is differentiable at 0 and that $f(0) = 0$. Prove that $f(x) = xg(x)$ for some function g which is continuous at 0.*

Solution. Define $g(x) = f(x)/x$ for $x \neq 0$ and $g(0) = f'(0)$, then, since $f(0) = 0$, we have $f(x) = xg(x)$ for all x. The function $g(x) = f(x)/x$ is continuous on $\mathrm{Dom}(f) \setminus \{0\}$; since

$$g(0) = f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} g(x),$$

g is continuous at 0 as well.

Chapter 11, pp. 206-211:

10pt **11.** *Among all circular cylinders of volume V find the one with the smallest surface area.*

Solution. Given $V > 0$, define the function $S(r) = 2\pi r V/(\pi r^2) + 2\pi r^2 = 2V/r + 2\pi r^2$, $r > 0$, then $S(r)$ is the surface area of the cylinder of volume V and base radius r. Since S is continuous and $S(r) \longrightarrow +\infty$ as $r \longrightarrow 0^+$ or $r \longrightarrow +\infty$, S attains its minimum value at some point $r_0 \in (0, +\infty)$, which must be a critical point for S. We have $S'(r) = -2V/r^2 + 4\pi r$, so $S'(r) = 0$ iff $4\pi r = 2V/r^2$ iff $r^3 = V/(2\pi)$ iff $r = \sqrt[3]{V/(2\pi)}$. So, the cylinder of the smallest surface area is the one with $r = \sqrt[3]{V/(2\pi)}$ and $h = V/(\pi r^2) = V/\big(\pi \sqrt[3]{V^2/(4\pi^2)}\big) = \sqrt[3]{4V/\pi} = 2\sqrt[3]{V/(2\pi)} = 2r$.

5pt **A1.** (a) *Prove that if f is convex on an open interval I and g is increasing and convex on $f(I)$, then $g\circ f$ is convex on I.*

Solution. First of all, f is continuous on I and $f(I)$ is therefore an interval too. Let $x, z \in I$, $t \in [0, 1]$. Since f is convex, $f(tx + (1-t)z) \leq tf(x) + (1-t)f(z)$. Since g is increasing, this implies $g\big(f(tx + (1-t)z)\big) \leq g\big(tf(x) + (1-t)f(z)\big)$. And as g is convex, $g\big(tf(x) + (1-t)f(z)\big) \leq tg(f(x)) + (1-t)g(f(z))$. Hence, $g\big(f(tx + (1-t)z)\big) \leq tg(f(x)) + (1-t)g(f(z))$, so $g\circ f$ is convex.

10pt (b) *Let f be a convex function on an open interval I. Prove that if f is strictly decreasing on I, then its inverse f^{-1} is also convex on $f(I)$, and if f is strictly increasing on I, then f^{-1} is concave on $f(I)$.*

Solution. Let f be convex and strictly monotone on I, then f is invertible, continuous on I, and $J = f(I)$ is also an open interval.

Let $u, v \in J$ and $t \in [0, 1]$, let $x = f^{-1}(u)$ and $z = f^{-1}(v)$. If f be strictly decreasing on I, then $f(tx + (1-t)z) \leq tf(x) + (1-t)f(z) = tu + (1-t)v$. Since f^{-1} is strictly decreasing, this implies that $tx+(1-t)z = f^{-1}\big(f(tx+(1-t)z)\big) \geq f^{-1}(tu+(1-t)v)$, that is, $tf^{-1}(u)+(1-t)f^{-1}(v) \geq f^{-1}(tu+(1-t)v)$. Hence, f^{-1} is convex on J.

If f is strictly increasing on I, then for $u, v \in J$ and $t \in [0, 1]$ we have $tf^{-1}(u) + (1-t)f^{-1}(v) \leq f^{-1}(tu + (1-t)v)$, that is, f^{-1} is convcave on J.

Chapter 11, Appendix, p. 228:

10pt **8.** *Prove Jensen's inequality: If f is a convex function on an interval I and $p_1, \ldots, p_n$ are positive numbers such that $\sum_{i=1}^n p_i = 1$, prove that for any $x_1, \ldots, x_n \in I$, $f\big(\sum_{i=1}^n p_i x_i\big) \leq \sum_{i=1}^n p_i f(x_i)$.*

Solution. I'll use induction on n; for $n = 2$ the statement it true. Given $p_1, \ldots, p_{n+1} > 0$ with $\sum_{i=1}^{n+1} p_i = 1$, define $p = \sum_{i=1}^n p_i$ and $q_i = p_i/p$, $i = 1, \ldots, n$, then $q_i > 0$ for all i and $\sum_{i=1}^n q_i = \sum_{i=1}^n p_i/p = 1$, so we may assume by induction that

$$f\Big(\sum_{i=1}^n q_i x_i\Big) \leq \sum_{i=1}^n q_i f(x_i).$$

Further, $p, p_{n+1} > 0$ and $p + p_{n+1} = 1$, so for $x = \sum_{i=1}^n q_i x_i$ we have

$$f\big(px + p_{n+1}x_{n+1}\big) \leq pf(x) + p_{n+1}f(x_{n+1}) \leq p\sum_{i=1}^{n+1} q_i f(x_i) + p_{n+1}f(x_{n+1}) = \sum_{i=1}^{n+1} p_i f(x_i).$$

Since $px + p_{n+1}x_{n+1} = \sum_{i=1}^{n+1} p_i x_i$, we have the induction step.

5pt **A2.** *By A1, the (strictly increasing) function* $\log_2$ *is (strictly) concave. Use this fact and the preceding problem to prove the general arithmetic-geometric mean inequality: for any* n *and positive* $x_1, \ldots, x_n$, $\frac{x_1+\cdots+x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}$.

Solution. Since $\log_2$ is a concave function on $(0, +\infty)$, for any n and $x_1, \ldots, x_n > 0$ we have

$$\log\left(\tfrac{1}{n}\sum_{i=1}^n x_i\right) \geq \tfrac{1}{n}\sum_{i=1}^n \log_2 x_i = \tfrac{1}{n}\log_2 \prod_{i=1}^n x_i = \log_2\left(\prod_{i=1}^n x_i\right)^{1/n}.$$

Since $\log_2$ is increasing, this implies that $\frac{1}{n}\sum_{i=1}^n x_i \geq \left(\prod_{i=1}^n x_i\right)^{1/n} = \sqrt[n]{\prod_{i=1}^n x_i}$.