Math 4181H **Solutions to Homework 8**

5pt **A3.** *Using the definition (and not using the Fundamental Theorem of Calculus), prove that* $\int_0^1 x^2 = 1/3$.

Solution. Since x^2 is a continuous function, it is integrable. For any $n \in \mathbb{N}$ consider the partition $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ of $[0,1]$. We then have

$$U(x^2, P_n) = \sum_{i=1}^{n} \Big(\frac{i}{n}\Big)^2 \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6},$$

which tends to $\frac{1}{3}$ as $n \to \infty$. Since $\text{mesh}(P_n) \to 0$ as $n \to \infty$, $\int_0^1 x^2 = \lim_{n\to\infty} U(x^2, P_n) = \frac{1}{3}$.

Chapter 13, pp. 275-281:

4. *Let* $d \in \mathbb{N}$ *and* $0 < a < b$, *put* $f(x) = x^d$. *Find* $\int_a^b f$ *in the following way: Put* $c = b/a$.

5pt (a) *For each* $n \in \mathbb{N}$, *put* $t_i = ac^{i/n}$, $i = 0, 1, \ldots, n$, *and let* $P_n = \{t_0, \ldots, t_n\}$. *Prove that* $\text{mesh}(P_n) \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Solution. For each i, $\Delta t_i = t_i - t_{i-1} = ac^{i/n} - ac^{(i-1)/n} = ac^{(i-1)/n}(c^{1/n} - 1) < ac(c^{1/n} - 1)$. Since $c^{1/n} \longrightarrow 1$ as $n \longrightarrow \infty$, $ac(c^{1/n} - 1) \longrightarrow 0$, so $\text{mesh}(P_n) = \max\{\Delta t_i,\ i = 1, \ldots, n\} \longrightarrow 0$.

10pt (b) *Find* $U(f, P_n)$. *(That is, try to write* $U(f, P_n)$ *in a compact form that will help you in (c) below.)*

Solution. Since f is increasing, the maximum of f on each interval $[t_{i-1}, t_i]$ is reached at the point t_i. So,

$$U(f, P_n) = \sum_{i=1}^{n} t_i^d \Delta t_i = \sum_{i=1}^{n} a^d c^{id/n} ac^{i/n}(1 - c^{-1/n}) = a^{d+1}(1 - c^{-1/n}) \sum_{i=1}^{n} (c^{(d+1)/n})^i$$
$$= a^{d+1}(1 - c^{-1/n})c^{(d+1)/n} \frac{c^{d+1} - 1}{c^{(d+1)/n} - 1} = a^{d+1}(c^{d+1} - 1)c^{(d+1)/n} \frac{1 - c^{-1/n}}{c^{(d+1)/n} - 1} = (b^{d+1} - a^{d+1})c^{d/n} \frac{c^{1/n} - 1}{c^{(d+1)/n} - 1}.$$

5pt (c) *Conclude that* $\int_a^b x^d dx = \frac{1}{d+1}(b^{d+1} - a^{d+1})$.

Solution. Since f is continuous, it is integrable. As $n \to \infty$ we have $c^{1/n} \to 1$, so $c^{i/n} \to 1$ for each $i = 1, \ldots, d$. So, $c^{d/n} \longrightarrow 1$ and

$$\frac{c^{1/n} - 1}{c^{(d+1)/n} - 1} = \frac{c^{1/n} - 1}{(c^{1/n})^{d+1} - 1} = \frac{1}{1 + c^{1/n} + \ldots + c^{d/n}} \longrightarrow \frac{1}{d+1}.$$

Hence, $U(f, P_n) \longrightarrow \frac{1}{d+1}(b^{d+1} - a^{d+1})$. So, $\int_a^b f(x)\,dx = \frac{1}{d+1}(b^{d+1} - a^{d+1})$.

21. *Let* f *be strictly increasing and continuous on* $[c, d]$, *let* $a = f(c)$ *and* $b = f(d)$.

10pt (a) *If* $P = \{t_0, \ldots, t_n\}$ *is a partition of* $[a, b]$, *let* $P' = \{f^{-1}(t_0), \ldots, f^{-1}(t_n)\}$. *Prove that* $L(f^{-1}, P) + U(f, P') = bd - ac$.

Solution. For each i, let $x_i = f^{-1}(t_i)$. Since f and f^{-1} are increasing,

$$L(f^{-1}, P) = \sum_{i=1}^{n} f^{-1}(t_{i-1})(t_i - t_{i-1}) = \sum_{i=1}^{n} x_{i-1}t_i - \sum_{i=1}^{n} x_{i-1}t_{i-1}$$

and

$$U(f, P') = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} t_i x_i - \sum_{i=1}^{n} t_i x_{i-1}.$$

Adding these two equalities, we get

$$L(f^{-1}, P) + U(f, P') = -\sum_{i=1}^{n} x_{i-1}t_{i-1} + \sum_{i=1}^{n} t_i x_i = -x_0 t_0 + x_n t_n = -ac + bd.$$

5pt (b) *Prove that* $\int_a^b f^{-1}(y)dy = bd - ac - \int_c^d f(x)\,dx$.

Solution. Since f and f^{-1} are monotone, they are integrable. By (a),

$$\int_a^b f^{-1} = L(f^{-1}) = \sup\{L(f^{-1},P) : P \text{ is a partition of } [a,b]\}$$
$$= \sup\{bd - ac - U(f,P') : P' \text{ is a partition of } [c,d]\}$$
$$= bd - ac - \inf\{U(f,P') : P' \text{ is a partition of } [c,d]\} = bd - ac - U(f) = bd - ac - \int_c^d f.$$

5pt (c) *Find* $\int_a^b \sqrt[n]{x}\,dx$ *for* $0 \le a < b$ *(without using the F.T.C.)*

Solution. By (b), applied to the function $f(x) = x^n$,

$$\int_a^b \sqrt[n]{x}\,dx = b\sqrt[n]{b} - a\sqrt[n]{a} - \int_{\sqrt[n]{a}}^{\sqrt[n]{b}} x^n\,dx = b^{1+1/n} - a^{1+1/n} - \tfrac{1}{n+1}\big((\sqrt[n]{b})^{n+1} - (\sqrt[n]{a})^{n+1}\big) = \tfrac{n}{n+1}(b^{1+1/n} - a^{1+1/n}).$$

31. *Let f be integrable on $[a,b]$.*

5pt (a) *Give an example where $f \ge 0$, $f(x) > 0$ for some $x \in [a,b]$, but $\int_a^b f(x)\,dx = 0$.*

Solution. $[a,b] = [-1,1]$, $f(x) = 0$ if $x \ne 0$ and $f(0) = 1$.

10pt (b) *Suppose that $f \ge 0$, f is continuous at $x_0 \in [a,b]$ and $f(x_0) > 0$. Prove that $\int_a^b f(x)\,dx > 0$.*

Solution. Let us assume that $x_0 \in (a,b)$; the cases $x_0 = a$ and $x_0 = b$ are similar. Let $f(x_0) = c > 0$. Find $\delta > 0$ such that $a < x_0 - \delta$, $x_0 + \delta < b$, and $f(x) > c/2$ for $x \in (x_0 - \delta, x_0 + \delta)$. Then

$$\int_a^b f(x)\,dx = \int_a^{x_0-\delta} f(x)\,dx + \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx + \int_{x_0+\delta}^b f(x)\,dx,$$

$\int_a^{x_0-\delta} f(x)\,dx \ge 0$, $\int_{x_0+\delta}^b f(x)\,dx \ge 0$, and $\int_{x_0-\delta}^{x_0+\delta} f(x)\,dx \ge (c/2)2\delta > 0$. So, $\int_a^b f(x)\,dx > 0$.

Chapter 14, pp. 296-302:

1. *Find the derivatives of the following functions:*

5pt (i) $F(x) = \int_0^{x^3} \sin^3 t\,dt$.

Solution. $F(x) = G(x^3)$ where G is the integral, and so, by the F.T.C., the primitive function of $\sin^3 x$. So, $F'(x) = G'(x^3)\cdot 3x^2 = \sin^3(x^3)\cdot 3x^2$.

5pt (ii) $F(x) = \int_3^{\int_1^x \sin^3 t dt} \frac{dt}{1+\sin^6 t + t^2}$.

Solution. $F(x) = G(\varphi(x))$ where G is a primitive of $\frac{1}{1+\sin^6 t+t^2}$ and $\varphi(x) = \int_1^x \sin^3 t dt$. So, $F'(x) = G'(\varphi(x))\varphi'(x) = \dfrac{1}{1+\sin^6(\varphi(x)) + \varphi(x)^2}\sin^3 x$. (If needed, φ can be computed to be $\varphi(x) = \frac{1}{3}\cos^3 x - \cos x - \frac{1}{3}\cos^3 1 + \cos 1$.)

10pt **9.** *Prove that if f is continuous on $\mathbb{R}$, then $\int_0^x f(u)(x-u)\,du = \int_0^x \big(\int_0^u f(t)\,dt\big)\,du$.*

Solution. Let $G(x) = \int_0^x f(u)(x-u)\,du$, $x \in \mathbb{R}$, then $G(x) = x\int_0^x f(u)\,du - \int_0^x f(u)u\,du$, so, by the F.T.C.,

$$G'(x) = \int_0^x f(u)\,du + xf(x) - f(x)x = \int_0^x f(u)\,du.$$

So, G is a primitive of the (continuous) function $F(x) = \int_0^x f(t)\,dt$, and since $G(0) = 0$, by the F.T.C. we have $G(x) = \int_0^x F(u)\,du = \int_0^x \big(\int_0^u f(t)\,dt\big)\,du$.

Another solution. Let $F(u) = \int_0^u f(t)\,dt$, since f is continuous, $F' = f$. Integrating by parts, we have $\int_0^x f(u)(x-u)\,du = \int_0^x (x-u)\,dF(u) = (x-u)F(u)\big|_0^x + \int_0^x F(u)\,du = \int_0^x \big(\int_0^u f(t)\,dt\big)\,du$.

5pt **21.** *Suppose that f' is integrable on $[0,1]$ and $f(0)=0$. Prove that for all $x \in (0,1]$ we have $|f(x)| \leq \sqrt{\int_0^x |f'|^2 dt}$.*

Solution. By the F.T.C., for any $x \in (0,1]$ we have $f(x) = \int_0^x f'(t)\,dt$, and by the Cauchy-Schwarz inequality,

$$\left|\int_0^x f'(t)\,dt\right| = \left|\int_0^x f'(t)\cdot 1\,dt\right| \leq \sqrt{\int_0^x f'(t)^2\,dt}\cdot\sqrt{\int_0^x 1^2\,dt} = \sqrt{\int_0^x f'(t)^2\,dt}\cdot\sqrt{x} \leq \sqrt{\int_0^x f'(t)^2\,dt}.$$